Improved bounds for the crossing numbers of $K_{m,n}$ and K_n

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Abstract

It has been long–conjectured that the crossing number $\mathrm{cr}(K_{m,n})$ of the complete bipartite graph $K_{m,n}$ equals the Zarankiewicz Number $Z(m,n) := \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{m}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor$. Another long–standing conjecture states that the crossing number $\mathrm{cr}(K_n)$ of the complete graph K_n equals $Z(n) := \frac{1}{4} \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor \lfloor \frac{n-2}{2} \rfloor \lfloor \frac{n-3}{2} \rfloor$. In this paper we show the following improved bounds on the asymptotic ratios of these crossing numbers and their conjectured values:

(i) for each fixed $m \geq 9$, $\lim_{n\to\infty} \mathrm{cr}(K_{m,n})/Z(m,n) \geq 0.83m/(m-1)$;

(ii) $\lim_{n\to\infty} \mathrm{cr}(K_{n,n})/Z(n,n) \geq 0.83$; and

(iii) $\lim_{n\to\infty} \mathrm{cr}(K_n)/Z(n) \geq 0.83$.

The previous best known lower bounds were $0.8m/(m-1), 0.8$, and 0.8, respectively. These improved bounds are obtained as a consequence of the new bound $\mathrm{cr}(K_{7,n}) \geq 2.1796n^2 - 4.5n$. To obtain this improved lower bound for $\mathrm{cr}(K_{7,n})$, we use some elementary topological facts on drawings of $K_{2,7}$ to set up a quadratic program on 6! variables whose minimum p satisfies $\mathrm{cr}(K_{7,n}) \geq (p/2)n^2 - 4.5n$, and then use state–of–the–art quadratic optimization techniques combined with a bit of invariant theory of permutation groups to show that $p \geq 4.3593$.

Keywords: crossing number, semidefinite programming, copositive cone, invariants and centralizer rings of permutation groups

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