# Local Systems and Derived Algebraic Geometry

Harrison Chen

March 10, 2015

## Contents



### Introduction

First, a warning. What we call the moduli stack of local systems in this note $\mathrm{Loc}_G(X)$ is called the moduli stack of $G$-bundles $\mathrm{Bun}_G(X)$ in the literature, e.g. in [AG:Sing]. In this paper the two notions coincide since for us $X$ will always be a topological space realized as a locally constant higher stack. However in general one may take $X$ to be, say, a scheme, which contains "infinitesimal data" to which Loc is sensitive to but Bun is not.

A note on references: Preygel's note [Pr:Stacks] was very useful for me in understanding the formalism surrounding (higher) stacks. Some definitions and discussions can be found in [AG:Sing] and [BN:Loop]. Some statements and examples were found on MathOverflow. Some additions are due to comments from audience members during the talk and conversations with David Nadler. For a survey on derived algebraic geometry, see [To:DAG]. These notes were written with a faulty memory and very incomplete understanding on my part. I expect there will be errors, and corrections are appreciated.

## 1 Classical notions of local systems

**Definition 1** (A $GL_n$-local system on a space)**.** Let $X$ be a topological space. There are three things which one might mean by a *local system*:
(a) ["Betti" definition] a vector bundle $\pi : E \to X$ with parallel transport, i.e. for each homotopy class of paths in $X$, a vector space isomorphism between fibers, respecting composition
(b) ["deRahm definition"] if $X$ is a differentiable manifold, a vector bundle $\pi : E \to X$ with a flat connection, i.e. a map of bundles $\nabla : E \to E \otimes T^*X$ satisfying the Leibniz rule $\nabla$ and such that $[\nabla_Y, \nabla_Z] = 0$ for vector fields $Y, Z$ (one can also write $\nabla : TX \to \mathrm{End}_X(E)$)
(c) [sheaf definition] a locally constant sheaf of vector spaces on $X$

*Sketch proof.* Definition (a) implies (b) by defining $\nabla_Y$ at a point $x \in X$ as follows: choose a path $\gamma$ starting at $x$ with tangent vector given by $Y$ and choose a local trivialization; differentiating the isomorphism gives an endomorphism of the fiber. Definition (b) implies (c) by locally solving the differential equations and gluing. Definition (c) implies (a) by the "expand and contract opens" argument: by compactness of $[0, 1]$, a path in $X$ can be covered by finitely many opens on which the sheaf is constant. Then, choosing points sequentially in the intersection, one gets a "zig-zag" of isomorphisms from $\gamma(0)$ to $\gamma(1)$. □

**Remark 2.** The equivalences (b) and (c) are essentially a special case of the Riemann-Hilbert correspondence. The full Riemann-Hilbert generalizes (b) to D-modules and (c) to constuctible sheaves.

**Example 3** (Local systems on the circle)**.** Let $X = S^1$. Fix a vector space $V$ and assign it to every point. Choose a point $x \in X$. For every other point in $y \in X$, choose once and for all a path $\gamma_y : x \to y$ and take the identity map $V \to V$ to correspond to this path. Further, choose once and for all a path $\alpha : x \to x$ that goes around the circle once. Every path in $S^1$ from $x \to y$ is homotopy equivalent to the composition of some number of $\gamma$ (or its inverse) and $\gamma_y$. Thus, to give a local system is to give an endomorphism corresponding to $\gamma$, i.e. an element of $GL(V)$. Note that this is equivalent to giving a map of groups $\pi_1(S^1) = \mathbb{Z} \to GL(V)$.

**Definition 4** (Group-y definition)**.** Let $X$ be a topological space. There are three things which one might mean by a *G-local system*:
(a) a principal $G$-bundle on $X$ with parallel transport, i.e. for each homotopy class of paths in $X$, an isomorphism of fibers which is $G$-equivariant, respecting composition
(b) if $X$ is a differentiable manifold, a principal $G$-bundle $\pi : E \to X$ with a flat connection, i.e. a map of Lie algebroids $\nabla : TX \to \mathfrak{e}$ such that $[\nabla_X, \nabla_Y] = 0$ (where $\mathfrak{e}$ is the associated Lie algebroid to the bundle $E$; note that the Leibniz rule translates into the Jacobi identity on $TX$)
(c) a locally constant sheaf of sets on which the constant sheaf $G$ acts transitively over $X$

*Proof of equivalence for $GL_n$.* The following gives a functorial correspondence between vector bundles and $G = GL_n$ bundles. Let $V$ be the standard representation of $GL_n$; to a $GL_n$-torsor $P$ we can associate the vector bundle $P \times_{GL_n} V$ using the Borel construction. In the other direction, for a vector bundle $E$ over $X$, one takes the frame bundle or the trivialization bundle. Parallel transport follows by functoriality. □

**Remark 5** (Groups other than $GL_n$)**.** Note that for groups other than $GL_n$, the above correspondence is not as obvious. In general, one could proceed by finding a faithful representation $V$ of $G$ and atempt to define an inverse functor to the Borel construction functor $P \mapsto P \times_G V$. To do so, we need to furnish $V$ with extra structure so that we can describe $G \subset GL(V)$.

For example, $G = SL_n$ torsors correspond to vector bundles with the extra data of a trivialization of their top exterior powers (their determinant bundles). That is, the inverse functor takes a rank $n$ vector bundle $E \to X$ with determinant $\det : \Lambda^n E \simeq \mathbb{C} \times X$ and associates to it its torsor of linear automorphisms whose top exterior power is the identity under det. For $G = SO_n$, the vector bundles should be equipped with the extra data of an inner product.

For $G$ semisimple and adjoint type (i.e. with trivial center, e.g. $PGL_2$), the adjoint representation is faithful, so one can form $P \times_G \mathfrak{g}$. We need to have some extra data on this resulting vector bundle in order to recover the group $G$ as well as the bundle $P$. It suffices to remember the inner product given by the Killing form, as well as the Lie bracket. In particular, with this structure we can choose any root datum and thus obtain a map of groups $\mathrm{Aut}(\mathfrak{g}) \to \mathrm{Aut}(D)$ where $D$ is the corresponding Dynkin diagram (note that we need to know the Lie bracket to talk about Aut and the Killing form to identify root data), and we can recover $G$ as the kernel of this map.

**Remark 6** (Alternative: Tannkian correspondence)**.** One can also see this correspondence in a somewhat formal way. Given a $G$-bundle $P$ on $X$, we associate to it a symmetric monoidal functor

$$\mathrm{Rep}(G) = \mathrm{QCoh}(BG) \to \mathrm{QCoh}(X)$$

given by the same construction, i.e. $V \mapsto P \times_G V$. Then one has defines a local system to be such a functor which is continuous, right t-exact, and sends flat objects to flat objects. See [AG:Sing] section 10.2 for details.

**Remark 7.** Finally, we observe that one can write the definition of a $G$-local system more invariantly by

$$\mathrm{Map}(\Pi_1(X), BG)$$

By writing this I mean, for each point of $X$, assign a set with a $G$-action, and for each path, a map of such sets intertwining the action. In particular, for each loop, one gets an automorphism of the $G$-torsor over a point, i.e. an element of $G$.

## 2 The stack $BG$ (as a moduli space in algebraic geometry)

**Remark 8** (Functor of points)**.** The Yoneda lemma gives an embedding

$$X \mapsto h_X = \mathrm{Hom}_{\mathbf{Sch}}(-, X) : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Fun}(\mathbf{Sch}, \mathbf{Set})$$

In other words, knowing a scheme $X$ is the same as knowing all maps from all schemes into $X$. There is a problem, however, of characterizing functors $\mathbf{Sch} \to \mathbf{Set}$ which are representable by schemes $X$ which is not easy in generality. Often moduli problems are more easily stated this way, and objects can also be more easily written in a coordinate-free way, at the cost of being less explicit. This is the approach we will take in these notes.

**Example 9.** The functor $\Gamma(X) := \Gamma(X, \mathcal{O}_X)$ (i.e. takes global sections) is a sheaf of sets and is represented by a $\mathbb{A}^1$. The functor $\Gamma^\times(X) = \Gamma(X, \mathcal{O}_X)^\times$ is represented by $\mathbb{G}_m$.

**Definition 10.** A (1-)*stack* is a (lax 2-)functor[1]

$$X : \mathbf{Aff}^{\mathrm{op}} \to \mathbf{Grpd}$$

satisfying some kind of (etale, fppf) sheaf condition. Note that this is not a higher analogue of schemes but rather "sheaves of sets."

**Remark 11** (2-categories)**.** The category of groupoids **Grpd** is a 2-category, which one has to define appropriately. Whatever it is, it should be the full subcategory of the 2-category of categories **Cat** consisting of groupoids, i.e. categories whose morphisms are all invertible. More elaborately, I will just say[2] that this 2-category has objects which are groupoids, (1-)morphisms which are functors between them, and "2-morphisms" between 1-morphisms which are natural transformations. As one might be used to in category theory, the "correct" notion of equvalence is not for functors to have inverses on the nose but inverses "up to homotopy" by a natural isomorphism i.e. the usual notion of equivalence of categories.

We should then also define a functor of 2-categories appropriately, i.e. given a map $S' \to S$ one needs a pullback map of groupoids $X(S) \to X(S')$. Here what we mean by lax 2-functor is that $X(g) \circ X(f)$ and $X(g \circ f)$ need not be equal but naturally isomorphic.

**Remark 12** (Sheaf property)**.** The sheaf property of a functor is with respect to a choice of covers on the source category (e.g. of schemes), called a *Grothendieck topology*. Popular candidates are the Zariski (Zariski opens), étale, or fppf topologies[3]. I want to surpress this theory for the entire talk, opting instead to just say "locally." The easiest example to keep in mind is the Zariski topology, i.e. $U$ is a disjoint union of opens covering $X$, and $U \to X$ is the obvious map.

A nice formal way to write down the sheaf property for a functor $F$ is to say that a certain truncated simplicial diagram is limit diagram. However, this notion of limit must be consider in a proper higher categorical way. The "top" level of the simplicial diagram should be thought of as a condition that must be satisfied, while the other levels contain data that must be furnished.

Let us do an example of a sheaf of *sets*: the functor $h_X = \mathrm{Hom}_{\mathbf{Sch}}(-, X)$. The sheaf condition is: to define a function $S \to X$ we must provide it on an open cover of $U \to S$, and it must satisfy a cocycle condition on double intersections in that cover. Diagrammatically, the following is a colimit

$$h_X(U \times_S U) \rightrightarrows h_X(U) \longrightarrow h_X(S)$$

where $U$ is the disjoint union of the open cover of $S$. Note that each of the $h_X(-)$ here are *sets*, and that the map is uniquely determined by $h_X(U)$, and exists so long as the cocycle condition on $h_X(U \times_S U)$ holds.

One category-level up, i.e. for 2-sheaves, our motivating example is torsors. To give a torsor, we must provide a trivialization on an open cover of $S$, provide gluings (extra data!) on double intersections, and these gluings must

[1] See Question 3119 on MathOverflow for a discussion on this. For better or worse, I will ignore all such technical details.
[2] One can look in nLab for less wishy-washy "definitions."
[3] These other topologies are used, for instance, because the Zariski topology does not have enough opens for things like the $n$-fold cover of $\mathbb{G}_m$ to be locally trivial.

satisfy a cocycle condition on triple intersections. Let $BG$ be this functor; then the sheaf property says that one must have a "colimit" diagram (in an appropriate homotopical sense)

$$BG(U \times_S U \times_S U) \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} BG(U \times_S U) \rightrightarrows BG(U) \longrightarrow BG(S)$$

Note here that the $BG(-)$ are (1-)*categories*. The sheaf condition has one extra term in the simplicial complex and requires an appropriate notion of (co)limit for 2-categories.

**Definition 13** (The stack of $G$-torsors)**.** Let $G$ be an algebraic group[4]. The stack $BG$ assigns to a scheme $S$ the groupoid of right $S$-torsors over $BG$. Explicitly, a right torsor over $S$ is a scheme $P$ equipped with a right (algebraic) $G$-action, and a map $P \to S$ such that there is a "cover" $S' \to S$ such that the base change $P' := P \times_S S'$ is trivializable, i.e. there is a $G$-equivariant isomorphism $P' \simeq G \times S'$ where $G$ acts on itself by right multiplication. A map of torsors a $G$-equivariant $S$-map $P \to Q$. By definition, a torsor satisfies a cocycle condition on triple intersections so that it satisfies the 2-sheaf cocycle condition which we glossed over above.

**Example 14** (Torsors over a point)**.** It is instructive to consider the category of right $G$-torsors over a point, i.e. the category of sets with a transitive *right* $G$-action. Any two $G$-sets are isomorphic, and an isomorphism is determined by a choice of $x \in X$ and $y \in Y$. Given such a choice, the isomorphism defined is $f(xg) = yg$. We call the choice of a base point $x \in X$ a *trivialization* of the $G$-set $X$, because it also determines for us an isomorphism $X \simeq G$ sending $x$ to 1. Thus, another way to say the above is that there is a unique map between two trivialized $G$-sets which respects the trivialization.

Further, trivializing $X$ and $Y$ allows us to give an explicit description of the morphisms of $G$-sets $f : X \to Y$ (which we do not require to intertwine with the chosen trivializations). Morphisms must commute with the right $G$-action, and such morphisms are given by *left* multiplication by $g \in G$. Composition is given by

$$G \xrightarrow{g} G \xrightarrow{h} G, \qquad G \xrightarrow{gh} G$$

and so intertwining operators

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[\beta]{} & Y \end{array}$$

satisfy $g\beta = \alpha g$. Note that if we insist that the map $X \to Y$ intertwines with the trivialization, there is only one such map, since it must send $x$ to $y$.

**Example 15** (Groupoid in schemes)**.** A stack can be presented as the data of a groupoid in schemes. That is, we give two schemes $U_0, U_1$ (think of $U_1$ as the morphisms and $U_0$ as the objects) with two maps $U_1 \to U_0$ (corresponding to source and target). Note that for any scheme $S$, one can "take $S$ points" of this presentation (i.e. consider $\mathrm{Map}(S, U_1) \to \mathrm{Map}(S, U_0)$) to get a groupoid of sets, which is equivalent to the usual notion of a groupoid. Note that a special case of a groupoid in schemes is the *quotient stack*. That is, if $G$ acts on $X$, then $X/G$ is a groupoid of schemes by taking $U_0 = X$ and $U_1 = G \times X$, with the source map $s$ being the projection and the target map $t$ being the action map.

There is a sheafy problem here. Note that for any scheme $U_0$, an $S$-point, i.e. function $S \to U_0$, determines and is determined by any compatible maps on a cover of $S$. There's a problem here which is the resulting functor is not "sheafy." For example, take the groupoid associated to $\mathrm{pt}/G$. Its $S$-points is a groupoid with one object, so this functor would appear to represent an object which is "globally trivializable" instead of locally.

**Definition 16** (Ad-hoc definition of fiber product in 2-categories)**.** Let $X, Y, Z$ be 2-stacks with $f : X \to Z$ and $g : Y \to Z$. Then define the fiber product $(X \times_Z Y)(S)$ as the groupoid whose objects are $x \in X(S), y \in Y(S)$ and an isomorphism $\alpha : f(x) \to g(y)$. The morphisms are maps intertwining such data.

[4]One can make this definition for any geoemtric group stack.

**Definition 17** (Representibility)**.** First, a map of stacks $f : X \to Y$ is *representable by schemes* if for every $\eta \in Y(S)$ with $S$ a scheme, the pullback is a scheme:

$$\begin{array}{ccc} X \times_Y S & \longrightarrow & X \\ \big\downarrow & & \big\downarrow f \\ S & \xrightarrow[\eta]{} & Y \end{array}$$

For any property preserved by base change (e.g. smooth, flat), we say that a representable $f$ has that property if every base change to a scheme has that property.

**Remark 18** (Geometric (Artin) stack)**.** In general, stacks as we've defined them thus far don't have to look very geometric at all. One candidate for a "geometric stack" is *Artin stacks*, which come equipped with an *atlas* $U \to X$ which is surjective, smooth and representable, and whose diagonal map $X \to X \times X$ is also representable. The upshot of Artin stacks is one can present them as a groupoid of schemes. In particular, one can recover a groupoid in schemes as a presentation of the stack by taking

$$U \times_X U \rightrightarrows U \longrightarrow X$$

where the two maps are the two projections. If $X$ is an Artin stack, then $U \times_X U$ is also a scheme.[5]

**Example 19.** We claim that $BG = \mathrm{pt}/G$ is an Artin stack. To see this, let $U$ be the functor whose $S$-points are torsors over $S$ along with a choice of trivialization (in particular, the torsor is trivial). The morphisms will be morphisms of torsors which intertwine with the trivializations, and any two objects in this category are canonically isomorphic, i.e. $U(S)$ is a one-object category with only the identity morphism, i.e. $U = \mathrm{pt}$ is a scheme. Further, one has that $U \to BG$ is surjective, since surjectivity of sheaves only needs to be satisfied locally and every torsor is locally trivializable.

To identify the descent data, we need to compute $\mathrm{pt} \times_{BG} \mathrm{pt}$. The objects of this category are given by two torsors with two trivializations, which map two the same torsor. In other words, a torsor with two choices of trivialization. The morphisms are given by pairs of morphisms respecting the trivializations. Given two trivializations, there is a unique morphism given by $g \in G$ relating them, so the objects are given by a torsor, a trivialization, and a $g \in G$. Further, any two torsors with the same $g$ are canonically isomorphic, so one has that $(\mathrm{pt} \times_{BG} \mathrm{pt})(S)$ consists of maps $S \to G$, i.e. $\mathrm{pt} \times_{BG} \mathrm{pt} = G$.

**Example 20** (The fundamental groupoid)**.** Let $X$ be a topological space. We can consider $X$ as a (2-)stack by taking the constant functor (a "truncation" of a space)

$$S \mapsto \Pi_1(X)$$

and sheafifying it. Note that this is not an Artin stack.

**Remark 21** (Higher stacks, $\infty$-groupoids, and spaces)**.** One might consider trying to replace the category of groupoids with higher categories of $n$-groupoids to get a theory of higher stacks. I will avoid the technical details here since they become more involved and hope that arguments later remain convincing.

**Example 22** (Topological spaces as locally constant higher stacks)**.** Any "nice" topological space can be considered as a higher stack in analogy to the fundamental groupoid example above. In particular, let $X$ be a topological space which comes with a simplicial presentation. One can then take the locally constant functor to this simplicial presentation.

[5] The correct definition actually asks that all the maps I asked here to be representable by schemes, to actually be representable by something intermediate called *algebraic spaces*. An algebraic space is a sheaf of sets, i.e. a 0-stack, which has geometric properties analogous to the ones I've described here, where the maps are required to be representable by schemes. The reason for this intermediate category is that certain constructions fail to be representable by schemes but can be represented by algebraic spaces. Sometimes algebraic spaces are referred to as 0-Artin stacks.

# 3 The moduli stack of local systems

**Definition 23** (Internal Hom of stacks)**.** The notion of an internal mapping space of stacks is motivated by thinking of stacks as sheaves. Recall from ordinary sheaves on a topological space that one has the internal *sheaf hom*:

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) := \mathrm{Hom}_{\mathrm{sheaves}}(i^*\mathcal{F}, i^*\mathcal{G})$$

where $i : U \to X$ is the inclusion. Note that the Hom in sheaves has the structure of the objects in the target category of the sheaves, (e.g. sets, abelian groups) given by "levelwise maps." The kind of formalism in doing this exercise extends to sheaves of categories. We thus define the internal mapping space between stacks

$$\mathrm{Map}_{St}(X, Y)(S) = \mathrm{Map}_{St,S}(X \times S, Y \times S) = \mathrm{Map}_{St}(X \times S, Y)$$

where the last equality follows from universal property of products.

**Definition 24** (Local system)**.** Let $X$ be a locally constant stack associated to a simplicial presentation of a topological space. Define the *moduli stack of local systems* on $X$ by

$$\mathrm{Loc}_G(X) = \mathrm{Map}_{St}(X, BG)$$

We will see later that this definition can be improved. One reason we might not like it is that it doesn't actually see any of the topology of $X$ beyond the fundamental groupoid $\Pi_1(X)$ (i.e. it doesn't see any "higher simplices" in $X$).

**Remark 25** (Truncation by fundmental group)**.** The simple observation that $BG$ takes values in groupoids means that in fact

$$\mathrm{Loc}_G(X) = \mathrm{Map}_{St}(\Pi_1(X), BG)$$

To elaborate a little more, let's consider an analogy. Let $\mathbf{C}$ be a category and $S$ a set, considered a a 0-category. A functor from a 1-category to a 0-category sends every morphism to the "identity morphism" on objects, i.e. factors uniquely

$$\mathbf{C} \to \Pi_0(\mathbf{C}) \to S$$

and so

$$\mathrm{Fun}(\mathbf{C}, S) = \mathrm{Fun}(\Pi_0(\mathbf{C}), S)$$

Note that we can only do this for locally constant sheaves. Truncating $BG$ pointwise in the same way gives something that isn't even a sheaf of sets!

**Remark 26** (Hom can be computed pointwise for locally constant source)**.** Recall that sheafy Hom cannot be taken pointwise, i.e. $\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) \neq \mathrm{Hom}(\mathcal{F}(U), \mathcal{G}(U))$. However, in the case of a locally constant sheaf $\mathcal{F}$ with value $F$, one can use the sheafification-forgetful adjunction to get

$$\mathrm{Hom}_{sh}(\mathcal{F}^s|_U, \mathcal{G}|_U) = \mathrm{Hom}_{presh}(F, \mathcal{G}|_U)$$

i.e. "sheafy hom can be computed pointwise." This will be useful to us in the following examples.

**Example 27** (A point)**.** The easiest example is:

$$\mathrm{Loc}_G(\mathrm{pt}) = \mathrm{Map}(\mathrm{pt}, BG) = BG$$

or more precisely, on $S$-points,

$$\mathrm{Loc}_G(\mathrm{pt})(S) = \mathrm{Map}(S, BG) = BG(S)$$

This agrees with the classical topological case, where a local system on a point is just a set with a $G$-action, or more generally for $S$-points, a $S$-torsor.

**Example 28** (A circle)**.** The next easiest example is

$$\mathrm{Loc}_G(S^1) = G/G$$

where we consider $S^1$ as a locally constant functor which takes values in the groupoid $S^1$. It may be useful to review Example 15. We present $S^1$ as a single 0-simplex and a single 1-simplex (with free compositions); let $X$ denote the sheafification of the constant sheaf with value the afore described category. There is a natural map $X \to \Pi_1(S^1)$ which is a isomorphism of sheaves, so our presentation is sufficient.

By our previous discussion of torsors, we take as an atlas for $\mathrm{Loc}_G(S^1)$ the category consisting of a torsor, a trivialization, and an automorphism (not necessarily respecting the trivialization). This is just the scheme $G$. The descent data is a category whose objects are a torsor, two automorphisms, and an intertwining operator, i.e. for automorphism $\alpha, \beta$, a $g$ such that $g\beta = \alpha g$, so this category is $G \times G \simeq \{(\alpha, \beta) \mid g\beta = \alpha g\}$.

In other words we have

$$G \times G \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} G \longrightarrow \mathrm{Loc}_G(S^1)$$

where $\pi_1(\alpha, \beta) = \alpha$ and $\pi_2(\alpha, \beta) = \beta$. Recall that in such descent diagrams which come from a group action on a scheme, $\pi_1$ is a projection to the scheme and $\pi_2$ is the action map. If we apply this interpretation here, one has that the action map $\pi_2$ sends $\alpha \mapsto \beta = g^{-1}\alpha g$. Thus, we find that $\mathrm{Loc}_G(S^1)$ is the adjoint quotient (i.e. where $G$ acts on itself by conjugation):

$$\mathrm{Loc}_G(S^1) = G/G$$

Note that one can also present $S^1$ using two 0-simplices and two 1-simplices connecting them, i.e.

$$S^1 = D^1 \coprod_{S^0} D^1$$

thus giving

$$\mathrm{Loc}_G(S^1) = \mathrm{Map}(S^1, BG) = BG \times_{BG \times BG} BG$$

which may be familiar to algebraic geometers as the inertia stack. This presentation has the psychological benefit that it can be written as some kind of (fiber) product; in some cases, when we need to derive the fiber product (which will be discussed later), this presentation also has computational benefits. Finally, observe that, using this presentation, $BG \to BG \times BG$ is flat, in fact a fibration with smooth fiber $G$. This is important; it means that the nonderived fiber product is "already derived."

**Remark 29** (Loop spaces)**.** The derived stack $\mathrm{Map}_{DSt}(S^1, X)$ is also known as the *free loop space* of $X$, denoted $\mathcal{L}X$. In homotopy theory we replace our stacky quotients with homotopy quotients, so $\mathcal{L}(BG)$ is the homotopy quotient $G/G$. If $G$ is discrete, this is the disjoint union of classifying spaces of stabilizers of orbits of $G$ (but in general it may be more complicated). To get the *based loop space* $\Omega X$, one takes a (derived/homotopy) fiber product (where $* \in S^1$ and $x \in X$ are the base points)

$$\begin{array}{ccc} \Omega X & \longrightarrow & \mathcal{L}X \\ \downarrow & & \downarrow{\scriptstyle \mathrm{ev}_*} \\ \mathrm{pt} & \underset{x \in X}{\longrightarrow} & X \end{array}$$

For $X = BG$ this is the base change

$$\begin{array}{ccc} G & \longrightarrow & G/G \\ \downarrow & & \downarrow{\scriptstyle \mathrm{ev}_*} \\ \mathrm{pt} & \underset{x \in X}{\longrightarrow} & \mathrm{pt}/G \end{array}$$

**Example 30** (The 2-sphere)**.** If we try to do the same with

$$\mathrm{Loc}_G(S^2)$$

we run into a "problem." First is that the fundamental groupoid of $S^2$, i.e. presented as a 1-category where we "forget" the 2-morphisms (and make the into identifications) is equivalent to a trivial category, and we find that $\mathrm{Loc}_G(S^2) = BG$. Passing to higher stacks does not help the situation, but let's do so anyway. Let $S^2$ be the locally constant stack with value the 2-groupoid presented by a single 0-simplex, a single 1-simplex, and two 2-simplices. In other words, we glue two disks along the circle $S^2 = D^2 \coprod_{S^1} D^2$. Thus we expect $\mathrm{Loc}_G(S^2) = BG \times_{G/G} BG$:

$$\begin{array}{ccc} ?? & \longrightarrow & e/G \\ \downarrow & & \downarrow \\ e/G & \longrightarrow & G/G \end{array}$$

Taking the usual pullback we get again $BG$. The pullback does not differentiate between imposing the same equation once or twice. We get the same result if we pass to higher stacks, since $BG$ is a 1-stack to begin with. One reason why we might expect something different is that in topology, pointed maps

$$\mathrm{Map}_*(S^2, X) = \mathrm{Map}_*(S^1 \# S^1, X) = \mathrm{Map}_*(S^1, \mathrm{Map}_*(S^1, X)) = \Omega^2 X$$

is the double pointed loop space of $X$. Thus we expect

$$\mathrm{pt} \times_{BG} \mathrm{Loc}_G(S^2) = \mathrm{Map}_*(S^2, BG) = \Omega^2(BG) = \Omega(G),$$

the loop space of $G$. Though we might not really expect something exactly like this, it should be some kind of algebraic analogue instead of the rather trivial based $\mathrm{Map}_*(S^2, X) = \mathrm{pt}$ (to computed pointed mapping stacks from mapping stacks, base change away from $BG$). We will discuss shortly that the problem is that the above fiber product should be derived in an appropriate sense.

**Example 31** (The wedge product of two $S^1$)**.** Let us take the wedge product of two $S^1$, i.e. $S^1 \coprod_{\mathrm{pt}} S^1$. This is

$$G/G \times_{BG} G/G = (G \times G)/G$$

where $G$ acts by the simultaneous adjoint action, by a similar computation.

**Example 32** (The 2-torus, commuting stacks)**.** Now, let us glue a 2-cell onto the wedge of two spheres

$$\begin{array}{ccc} S^1 & \longrightarrow & S^1 \wedge S^1 \\ \downarrow & & \downarrow \\ D^2 & \longrightarrow & T \end{array}$$

to get the 2-torus and compute

$$\mathrm{Loc}_G(S^1 \times S^1)$$

One can check that the equation imposed by the 2-cell is the commutator, and this results in the pullback diagram

$$\begin{array}{ccc} \mathrm{Loc}_G(T) & \longrightarrow & (G \times G)/G \\ \downarrow & & \downarrow{\scriptstyle [-,-]} \\ e/G & \longrightarrow & G/G \end{array}$$

This is the *commuting stack* of $G$. In general this fiber product should also be derived, though unlike in the case of $S^2$, the classical commuting stack is an interesting object. Note that one has similarly that the derived mapping spaces

$$\mathrm{Map}_{DSt}(T, X) = \mathrm{Map}_{DSt}(S^1 \times S^1, X) = \mathrm{Map}_{DSt}(S^1, \mathrm{Map}_{DSt}(S^1, X)) = \mathcal{L}\mathcal{L}X$$

i.e. $\mathrm{Loc}_G(T)$ is the the double free loop space of $BG$.

# 4 Derived algebraic geometry

## 4.1 Motivation: base change, convolution of integral kernels

**Definition 33** (Integral kernels)**.** Let $X$ and $Y$ be two proper schemes, and let $\mathcal{K} \in \mathrm{DCoh}(X \times Y)$. Then, $\mathcal{K}$ defines a functor $\Phi_{\mathcal{K}} : \mathrm{DCoh}(X) \to \mathrm{DCoh}(Y)$ by the formula

$$F(\mathcal{M}) = (\pi_Y)_*(\pi_X^*\mathcal{M} \otimes_{\mathcal{O}_{X\times Y}} \mathcal{K})$$

where all functors are derived. The sheaf $\mathcal{K}$ is called an *integral kernel.* Note we required properness so that the derived pushforward lands in coherent sheaves; if we are willing to work in quasicoherent sheaves this is not necessary.

**Remark 34** (Actions on categories)**.** In the above definition, if we let $Y = X$, then what we have is for an object of $\mathrm{DCoh}(X \times X)$ and an object of $\mathrm{DCoh}(X)$, the functorial assignment of another object of $\mathrm{DCoh}(X)$. This looks like we can say something like "there is an action of $\mathrm{DCoh}(X \times X)$ on $\mathrm{DCoh}(X)$" but it is not clear yet that there is a good notion of composition of two integral kernels that makes this action associative. In showing that this composition (by convolution) exists we need to use base change.[6] The following discussion is mostly formal in nature.

**Proposition 35** (Base change for flat morphisms)**.** *Let $f : X \to Y$ be any morphism and $g : Y' \to Y$ a flat morphism of schemes. Consider the diagram (with $X' := Y' \times_Y X$):*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

*Then one has a natural equivalence of functors (all derived appropriately)*

$$g^* f_* \simeq (f')_*(g')^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y')$$

**Proposition 36** (The projection formula)**.** *Let $f : X \to Y$ be a map of schemes, $\mathcal{E}$ a locally free sheaf on $Y$, and $\mathcal{F}$ be a quasicoherent sheaf. Then there is a natural isomorphism*

$$f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E}) \cong f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{E}$$

*where all functors are derived (though $f^*$ need not be, $\mathcal{E}$ being locally free). If we derive the tensor products and the pullback, then we don't need to assume $\mathcal{E}$ locally free.*

*Proof.* This essentially follows formally from base change. Let $E = \mathrm{Spec}_{\mathcal{O}_Y} \mathrm{Sym}_{\mathcal{O}_Y} \mathcal{E}$ and consider

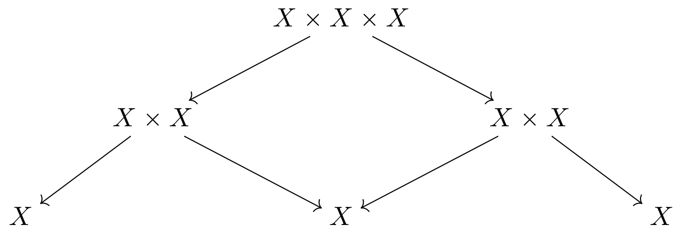
$$\begin{array}{ccc} X \times_Y E & \longrightarrow & X \\ \downarrow & & \downarrow \\ E & \longrightarrow & Y \end{array}$$

Note that $X \times_Y E = \mathrm{Spec}_{\mathcal{O}_X} \mathrm{Sym}_{\mathcal{O}_X} f^*\mathcal{E}$. Do base change, and take first homogeneous degree. □

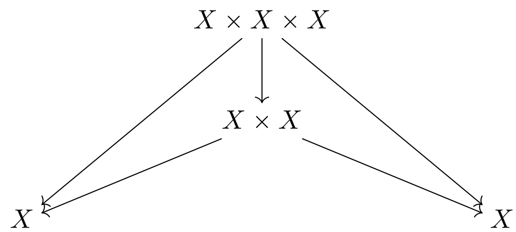
**Remark 37** (Composition of integral functors is convolution of integral kernels)**.** Now, let $X = Y$ and let $\mathcal{K}_1, \mathcal{K}_2 \in \mathrm{DCoh}(X \times X)$. I claim that the composition of the functors $\Phi_{\mathcal{K}_1} \circ \Phi_{\mathcal{K}_2}$ is given by the convolution of the two integral

[6] In general, for some variety $Z$ equipped with two flat "projection" maps $\pi_i : Z \to X$, the same formalism gives an action of $\mathrm{DCoh}(Z)$ on $\mathrm{DCoh}(X)$.

kernels. The proof uses base change and the projection formula. More precisely




Composition is pushing and pulling along the bottom, whist tensoring with the kernels on $X \times X$. By base change insteading of pulshing and pulling from the bottom middle, we can do it on $X \times X \times X$ which is the pullback of the center square. Since pullback and tensors commute, equivalently we can pull back both $\mathcal{K}_1$ and $\mathcal{K}_2$ to $X^3$ and then tensor, then push to $X \times X$, tensor with $\mathcal{K}_1$, and then push to $X$. By the projection formula, we can instead pull $\mathcal{M}$ to $X^3$, tensor with $\pi_1^* \mathcal{K}_1 \otimes \pi_2^* \mathcal{K}_2$, and then push back to $X$. Thus we have, effectively,




By the projection formula again, we can instead pull $\mathcal{M}$ to $X \times X$, tensor with the convolution of the kernels, and then push to $X$.

**Example 38** (Satake correspondence)**.** Recall now that for $B \backslash G/B$ acts on $(G/B)/G$ by convolution in a similar way, that is one has the two projections $G/B \times G/B \to G/B$, and then taking a quotient by $G$. The above arguments still apply since the projection map is smooth with fiber $G/B$, so it is flat, and so we have base change.

**Example 39** (Geometric Satake correspondence, failure of base change)**.** Sometimes we will want to do the same thing when the map is not flat. Recall that $\mathrm{Loc}_G(S^2) = (e \times_G e)/G$, which one might expect to act on $BG$. However, the map $e \to G$ is not flat, so our formalism above fails. Let us consider instead the following analogous example to see what goes wrong in base change. Let $f : \{0\} \to \mathbb{A}^1$ be the inclusion of the origin into the affine line, i.e. on rings the quotient map $k[x] \to k[x]/(x) \cong k$. It's not hard to show that the fiber product is

$$\begin{array}{ccc} \{0\} & \longrightarrow & \{0\} \\ \downarrow & & \downarrow \\ \{0\} & \longrightarrow & \mathbb{A}^1 \end{array}$$

However this diagram does not satisfy base change on derived categories, since when computing the derived pullback of the skyscraper sheaf we must resolve the $k[x]$-module $k$ by a projective resolution

$$k[x] \xrightarrow{x} k[x]$$

before tensoring $- \otimes_{k[x]} k$ to obtain the complex

$$k \xrightarrow{0} k$$

However, going the other way of course we have the usual skyscraper sheaf.

## 4.2 Resolving using dg algebras (in the affine case)

**Remark 40** (Proof of base change in affine case)**.** What went wrong here? Let's recall that proof of base change in the affine case. Take $X = \mathrm{Spec}(R), Y = \mathrm{Spec}(Q), Z = \mathrm{Spec}(k)$, then

$$\begin{array}{ccc} X \times_Z Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

base change says

$$M \otimes_k^L Q = M \otimes_R^L (R \otimes_k Q)$$

The problem is that one of these tensor products is not derived; if we were working with *modules* we could derive the last fiber product and everything would follow from some homological algebra. If $Q$ is flat over $k$ then there is no need to derive it. But otherwise, we need some way to resolve $Q$ over $k$ by free $k$-modules, but as an algebra. The solution is semifree dg algebras.

**Definition 41.** Let $M, N$ be two chain complexes, cohomologically graded (differentials increase degree). We define their tensor product by the usual "convolution" tensor product of graded modules:

$$(M \otimes N)_k := \bigoplus_{i+j=k} M_i \otimes N_j$$

and equip it with the differential (extended linearly):

$$d(m \otimes n) := d(m) \otimes n - (-1)^{\deg(m)} m \otimes d(n)$$

This defines a monoidal structure on chain complexes. A *dg algebra* over $k$ is a category enriched over chain complexes with a single object. More explicitly, it is a chain complex $A$ equipped with a multiplication $m : A \otimes A \to A$ which is a map of chain complexes. Even more explicitly, the multiplication should satisfy the rule:

$$d(x \cdot y) = d(x) \cdot y - (-1)^{\deg(x)} x \cdot d(y)$$

A dg algebra is *dg commutative* if $m(x, y) = m(y, x)$, i.e.

$$a \cdot b = (-1)^{\deg(a)} b \cdot a$$

A morphism of dg algebras is a functor of such categories.

**Example 42.** Every $k$ algebra is a dg algebra, concentrated in degree 0. It is dg commtuative if and only if it is commutative.

**Example 43.** Let us define a free dg algebra on one generator in odd degree -1. That is, $A = k[\lambda]$ for $\deg(\lambda) = -1$, with zero differential. Note that by dg commutativity, $\lambda \cdot \lambda = -\lambda \cdot \lambda$, so in characteristic not 2, $\lambda^2 = 0$. Thus $A$ is a two dimension vector space, with one dimension in degree 0 and one dimension in degree -1.

Now let us define a free dg algebra on two generators in degree -1. That is, let $A = k[\lambda_1, \lambda_2]$ with $\deg(\lambda_i) = -1$, with zero differential. Then likewise one has $\lambda_i^2 = 0$, but one has that $\lambda_1 \wedge \lambda_2 = -\lambda_2 \wedge \lambda_1$. So $A$ has dimension four, with one dimension vector $(1, 2, 1)$. So we see that in odd degree the multiplication behaves like an exterior product.

**Example 44.** Let us define a free dg algebra on one generator in degree -2. That is, let $A = k[\beta]$ with $\deg(\beta) = -2$, and zero differential. Then one has that $A$ is infinite dimensional, with one dimension in every nonpositive even dimension. Here the multiplication behaves like the usual multiplication in commmutative rings.

**Definition 45.** A dg algebra is *semifree* if the underlying graded algebra is free. A map of dg algebras is a quasi-isomorphism if it induces an isomorphism in cohomology.

**Example 46** (Koszul resolution)**.** Let $R = k[x_1, \dots, x_n]/(f_1, \dots, f_r)$. For each such $R$ we have a resolution over $S = k[x_1, \dots, x_n]$ called the *Koszul resolution.* Let $V^*$ be an $r$-dimensional $k$-vector space spanned by $df_i$. Then the Koszul resolution is given by

$$R \simeq S[V]$$

where $\deg(V) = -1$ and the differential is given by $d(df_i) = f_i$. Note that the rest of the differentials are given by the Leibniz rule, e.g.

$$d(df \wedge dg) = f\,dg - g\,df$$

**Example 47** (Self intersection of two origins)**.** Going back to our original example, one comutes that

$$\{0\} \times_{\mathbb{A}^1} \{0\} = k[\lambda]$$

with $\deg(\lambda) = -1$ and $d(\lambda) = 0$.

## 4.3 Some theory: what kind of category do dg algebras form?

**Remark 48** (Derived rings)**.** This subsection mostly serves to give an overview and references on the theory. What we've done above essentially is, when considering an $\infty$-stack as a functor $\mathbf{Rng} \to \infty - \mathbf{Grpd}$, we've replaced the category of rings with the category of dg algebras. There is a theoretically cleaner and more general category to replace rings with, which we will, with vagueness, call *derived rings.* Lurie in his thesis *Derived Algebraic Geometry* [Lu:Cr] discusses, along with dg algebras, two other candidates for derived rings: simplicial rings and $E_\infty$ ring spectra. In particular all three of these categories are ($\infty$, 1)-categories. It turns out that dg algebras are only well-behaved in characteristic zero. Here, the monoidal Dold-Kan correspondence gives some kind of equivalence between simplicial rings and dg algebras.

**Remark 49** (Morphisms in the category of dg algebras)**.** Dg algebras form an $(\infty, 1)$-category, though there is some delicacy required in correctly defining the morphisms of dg algebras and considering them as a "space." For a treatment, see the survey article [To:DAG].

**Remark 50** (Derived scheme)**.** Toen [To:DAG] gives a notion of a derived scheme in characteristic zero using dg schemes. An *affine derived scheme* is defined to be the "Spec" of a dg algebra. In general, a *derived scheme* is a scheme $X$ equipped with a sheaf of dg-algebras $\mathcal{O}_X^\bullet$ satisfying $H^0(X, \mathcal{O}_X^\bullet) = \mathcal{O}_X$ and $H^i(X, \mathcal{O}_X)$ quasicoherent over $\mathcal{O}_X$. The functor from derived schemes into derived stacks is not fully faithful, see [BN:Loop] Remark 3.3.

**Remark 51** (Derived stacks)**.** In the way that stacks replaced **Set** with a suitable model category or infinity category, one can do the same with $\mathbf{Rng} = \mathbf{Aff}^{\mathrm{op}}$. That is, a derived stack is a $(\infty, 1)$ functor

$$\mathbf{dgAlg}^{\mathrm{op}} \to \mathbf{Spaces}$$

satisfying a sheaf condition. Many details of the theory are being ignored in this exposition, for example what the analogous etale or fppf topology is on these categories, and good notions of Artin stack, et cetera. Note that an affine derived scheme (i.e. a dg-algebra) can be considered as a derived stack by an $(\infty, 1)$-version of the Yoneda lemma.

## 4.4 Local systems revisited

**Definition 52.** We can now revise our definition of local systems, where $X$ is a locally constant functor

$$\mathrm{Loc}_G(X) = \mathrm{Map}_{DSt}(X, BG)$$

where the mapping space is now taken in the category of derived stacks.

**Example 53** (Local systems on $S^2$)**.** From our above computations, one has

$$\mathrm{Loc}_G(S^2) = (\mathrm{Spec}\,\mathrm{Sym}_k(\mathfrak{g}^*[1]))/G$$

i.e. we use the Koszul resolution of the identity element of $G$, and tensoring with $\mathcal{O}_e$ kills all differentials. Note that if we forget the derived structure, we obtain again $\mathrm{Loc}_G(S^2)^{\mathrm{cl}} = \mathrm{pt}$.

**Remark 54.** Note that $S^2 = S^1 \# S^1$ (the smash product, which is the categorical product in the category of based spaces), and so we expect that (where Map denotes here based maps)

$$\mathrm{Map}_*(S^2, X) = \mathrm{Map}_*(S^1 \# S^1, X) = \mathrm{Map}_*(S^1, \mathrm{Map}_*(S^1, X)) = \Omega\Omega X,$$

the double based loop space.

Recall that $\Omega(BG) = G$. Then $\Omega(G)$ can be computed by

$$\begin{array}{ccc} \Omega G & \longrightarrow & G \times_{G \times G} G \\ \downarrow & & \downarrow{\scriptstyle \pi_1 = \pi_2} \\ \mathrm{pt} & \xrightarrow[e \in G]{} & G \end{array}$$

One computes that $\Omega^2(BG) = \mathrm{Spec}\,\mathrm{Sym}_k(\mathfrak{g}^*[1])$, whose classical scheme is a point. This is obtained from $\mathrm{Loc}_G(S^2)$ by affixing a base point, i.e. base changing away from $BG$:

$$\begin{array}{ccc} \Omega^2 BG \simeq \mathrm{Spec}\,\mathrm{Sym}_k(\mathfrak{g}^*[1]) & \longrightarrow & \mathrm{Loc}_G(S^2) \simeq \mathrm{Spec}\,\mathrm{Sym}_k(\mathfrak{g}^*[1])/G \\ \downarrow & & \downarrow{\scriptstyle \mathrm{ev}_*} \\ \mathrm{pt} & \xrightarrow[\mathrm{ev}_x]{} & BG \end{array}$$

**Remark 55.** This definition of local system differs from that in [AG:Sing], because we restrict our attention to the case when $X$ is a locally constant functor, where $\mathrm{Bun}_G$ and $\mathrm{Loc}_G$ are the same. These two diverge when one wants to consider more generally, schemes, Artin stacks, and derived stacks.

# References


[AG:Sing] Dima Arinkin, Dennis Gaitsgory, Singular support of coherent sheaves and the Geometric Langlands conjecture, arXiv:1201.6343v4 2014.

[BN:Loop] David Ben-Zvi, David Nadler, Loop Spaces and Connections, arXiv:1002.3636v2, 2011.

[Lu:Cr] Jacob Lurie, Derived Algebraic Geometry, Ph.D. thesis, 2004.

[Pr:Stacks] Anatoly Pregel, Notes on stacks.

[To:DAG] Bertrand Toën, Derived Algebraic Geometry, arXiv:1401.1044v2, 2014.