Math Jams Worksheet, Week 1

February 3rd, 2008

Easier Examples

1. Let $V = \mathbb{R}^2$. Show that the vectors $(1, 1)$ and $(2, 3)$ span V as an $\mathbb{R}$-vector space. Draw a picture showing the subspace of $\mathbb{R}^2$ spanned by $(1, 1)$. Is $\{(1, 1), (2, 3)\}$ a basis for $\mathbb{R}^2$?

2. Let $V = \mathbb{R}^3$. Let W be the subspace of V spanned by the vectors $(1, 1, 0)$, $(0, 0, 1)$, and $(1, 1, 1)$. Draw a picture showing this subspace. Find a basis for W. What is the dimension of W? Find a one-dimensional subspace W' of V such that $W + W' = \mathbb{R}^3$. Is this sum direct? Geometrically speaking, what needs to be true of W' so that $W + W' = \mathbb{R}^3$?

Harder Examples

1. Let $C^\infty(\mathbb{R})$ be the set of infinitely-differentiable real-valued functions on $\mathbb{R}$.

   a) Show that this set has the structure of a vector space over $\mathbb{R}$, where addition is given by the usual addition of functions, and scalar multiplication is likewise given as usual.

   b) Let $C_0^\infty(\mathbb{R})$ be the subset of $C^\infty(\mathbb{R})$ consisting of those functions such that $f(0) = 0$. Show that $C_0^\infty(\mathbb{R})$ is a vector subspace of $C^\infty(\mathbb{R})$. What is the dimension of this subspace?

   c) Find a one-dimensional subspace V of $C^\infty(\mathbb{R})$ such that $V + C_0^\infty(\mathbb{R}) = C^\infty(\mathbb{R})$. Is this sum direct?

2. Let $n > 0$ be a fixed natural number, and let $P_n(\mathbb{C})$ be the set of all polynomials $f(x)$ in one variable x with complex coefficients with degree less than or equal to n.

a) Show that $P_n(\mathbb{C})$ is a vector space over $\mathbb{C}$, with the usual operations of addition and scalar multiplication.

b) What is the dimension of $P_n(\mathbb{C})$ as a $\mathbb{C}$-vector space? Find a basis for $P_n(\mathbb{C})$ as a $\mathbb{C}$-vector space.

c) Show that $P_n(\mathbb{C})$ is also an $\mathbb{R}$-vector space, with the usual operations of addition and scalar multiplication. Give a basis for $P_n(\mathbb{C})$ as an $\mathbb{R}$-vector space.

3. (Practice with direct sums):

   a) Let V be a real vector space, and W and W' two vector subspaces of V with $\dim W = n < \infty$ and $\dim W' = m < \infty$. Show that the sum $W + W'$ is direct if and only if $\dim(W + W') = n + m$.

   b) Let V be the $\mathbb{C}$-vector subspace of $P_3(\mathbb{C})$ spanned by $\{1, x, x^2\}$, and let W be the subspace of $P_3(\mathbb{C})$ spanned by $\{x^2 + x^3, x^3\}$. Show that $V + W = P_3(\mathbb{C})$. Show moreover that this sum is *not* direct.

   c) Show that there is a bijection between (non-ordered) pairs of distinct lines through the origin in $\mathbb{R}^2$ and ways of writing $\mathbb{R}^2$ as a direct sum of two distinct one-dimensional subspaces (where again we ignore the order).

4. Let $f_0(x), \ldots, f_n(x)$ be $n + 1$ polynomials with real coefficients such that $\deg f_i(x) \leq n - 1$ for all i. Show that there exist real numbers $a_0, \ldots, a_n$, not all 0, such that

$$a_0 f_0(x) + \ldots + a_n f_n(x) = 0.$$

5. Let $P_{n,0}(\mathbb{R})$ be the set of polynomials $f(x)$ with real coefficients such that:

   i) $\deg f(x) \leq n$, and
   ii) $f(0) = 0$.

   Then show that:
   a) $P_{n,0}(\mathbb{R})$ is an $\mathbb{R}$-vector subspace of $P_n(\mathbb{R})$,
   b) $\dim P_{n,0}(\mathbb{R}) = n$. (Hint: find a set of n linearly independent vectors which spans $P_{n,0}$.)
   c) Notice that $P_n(\mathbb{R})$ is an $\mathbb{R}$-vector subspace of $C^\infty(\mathbb{R})$. Show that $P_{n,0}(\mathbb{R}) = P_n(\mathbb{R}) \cap C_0^\infty(\mathbb{R})$. Explain why this immediately shows part a) of this problem.

6. If you know about linear maps, show that the map $\phi_0 : P_n(\mathbb{R}) \to \mathbb{R}$ defined by $\phi_0(f) = f(0)$ is linear. If you know about kernels, what is the kernel of this map? Show that this map is surjective. If you know rank-nullity, explain how this gives another way of proving part b) of the previous problem.

7. Let $d : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ be the differentiation map, i.e. $d(f(x)) = f'(x)$.

   a) Show that d is a linear map.

   b) Consider the restriction of d to the subspace $P_5(\mathbb{R}) \subset C^\infty(\mathbb{R})$. Write a matrix for d with respect to the basis $\{1, x, x^2, \ldots, x^5\}$ of $P_5(\mathbb{R})$.

   c) Let A be the matrix you computed in part b). Compute $\det A$.

   d) Is A invertible? Why or why not? Give both a "linear algebra answer" and a "calculus answer."

8. (Harder): Let V be an n-dimensional vector space over $\mathbb{R}$, where $n < \infty$. Show that $V \cong \mathbb{R}^n$, that is, show that there are linear maps $\phi : V \to \mathbb{R}^n$ and $\psi : \mathbb{R}^n \to V$ such that $\phi \circ \psi$ and $\psi \circ \phi$ are the identity maps on $\mathbb{R}^n$ and V, respectively.