Math 74 Quiz 5 Solutions

October 12, 2008

1. (a) Define *countably infinite*.
   (b) Define *countable*.

Solution to a): A set X is called *countably infinite* if there exists a bijection $f : \mathbb{N} \to X$.

Solution to b): A set X is called *countable* if it is countably infinite or finite. (Alternatively, X is countably infinite if X is empty or there exists a surjection $f : \mathbb{N} \to X$. Note: I gave credit for this answer without the "is empty or" portion, since I forgot to mention this in office hours.)

Note of Concern: Too many people missed this question; remember that when we began the class I said I wouldn't fail anyone provided they made good efforts on all the homeworks, attended class, and showed me on the quizzes that they knew the definitions. Well, a lot of you didn't fulfill part three of this bargain on this quiz. Be concerned. If you don't know the basic definitions which are being used in everything we're doing in class, then you can't possibly understand any of the theorems that are built on these definitions.

2. Using the results proved in class, show that if X and Y are countable sets, so is $X \cup Y$.

Solution: Let X and Y be countable sets. If X is empty, then $X \cup Y = Y$ is countable, and similarly if Y is empty, so we can assume neither set is empty. Then there exist surjective functions $f : \mathbb{N} \to X$ and $g : \mathbb{N} \to Y$. Let $h : \mathbb{N} \to X \cup Y$ be the function defined by

$$h(n) = \begin{cases} f\left(\frac{n}{2}\right) & n \text{ even,} \\ g\left(\frac{n-1}{2}\right) & n \text{ odd.} \end{cases}$$

I claim that h is surjective. Indeed, if $z \in X \cup Y$, then $z \in X$ or $z \in Y$. If $z \in X$, then there is a number $m \in \mathbb{N}$ such that $f(m) = z$. Then $h(2m) = f(m) = z$. If $z \in Y$, then there is an $m \in \mathbb{N}$ such that $g(m) = z$. Then $h(2m+1) = g(m) = z$. Hence h is surjective, so $X \cup Y$ is countable.

Note: The whole idea to this solution is that if you can "count" X and Y, then you can count $X \cup Y$ by first "counting the first thing in X," then "counting the first thing in Y," then "counting the second thing in X," and so on.