Math 74 Quiz 4 Solutions

October 4, 2008

1. Explain what is wrong with the following proof that every natural number is greater than or equal to 2^{10}:

   Suppose it is false that every natural number is greater than or equal to 2^{10}, i.e. suppose there is some $n \in \mathbb{N}$ such that $n < 2^{10}$. Let $S = \{n \in \mathbb{N} \mid n < 2^{10}\}$. By assumption, $S \neq \emptyset$, so S has a smallest element n_0. Hence $n_0 - 1 \notin S$, so by definition of S, it follows that $n_0 - 1 \geq 2^{10}$. Hence $n_0 \geq 2^{10} + 1 \geq 2^{10}$, a contradiction. Thus our assumption was false and every natural number is greater than or equal to 2^{10}.

   Solution: Although it is correct to conclude that $n_0 - 1 \notin S$, we did not demonstrate that $n_0 - 1 \in \mathbb{N}$, so we cannot use this to conclude that $n_0 - 1 \geq 2^{10}$.

2. Carefully state the principle of inclusion-exclusion.

   Solution: Let X and Y be finite sets. Then $|X \cup Y| = |X| + |Y| - |X \cap Y|$.

   Note: A number of people forgot to say that X and Y needed to be finite. More worryingly, many people wrote nothing more than the formula $|X \cup Y| = |X| + |Y| - |X \cap Y|$, without ever saying what X and Y were! A mathematical statement involves words, not just formulas.

3. Define a function $f : \mathbb{N} \to \mathbb{Z}$ inductively as follows: let $f(0) = f(1) = 2$, and for $n \geq 2$, let $f(n) = f(n-1)f(n-2) - 1$.

   (a) Calculate $f(5)$.

   (b) Show that $f(n) \geq 2$ for all $n \in \mathbb{N}$.

Solution to a): Apply the inductive rule repeatedly:

$$\begin{aligned} f(0) &= 2 \\ f(1) &= 2 \\ f(2) &= 2 \cdot 2 - 1 = 3 \\ f(3) &= 3 \cdot 2 - 1 = 5 \\ f(4) &= 5 \cdot 3 - 1 = 14 \\ f(5) &= 14 \cdot 5 - 1 = 69. \end{aligned}$$

Solution to b): We prove this by induction on n. Let $P(n)$ be the statement "$f(k) \geq 2$ for all $k \in \mathbb{N}$ with $0 \leq k \leq n$."

Induction Beginning: $P(0)$ and $P(1)$ are automatic since by definition $f(0) = f(1) = 2$.

Induction Step: Let $n \geq 1$ and suppose $P(n)$ is true, i.e. suppose that $f(k) \geq 2$ for all $k \in \mathbb{N}$ with $0 \leq k \leq n$. We want to show that $P(n+1)$ is true, for which it will suffice to show that $f(n+1) \geq 2$. Now, by the induction hypothesis, we know that $f(n) \geq 2$ and $f(n-1) \geq 2$. Hence

$$f(n+1) = f(n)f(n-1) - 1 \geq 2 \cdot 2 - 1 = 3 > 2,$$

as desired.

Note: The trick of assuming that something holds for all numbers less than or equal to k (rather than just for k) is called *strong induction*, and is often useful.