

## Math 74 Midterm 2: Solutions

November 12, 2008

1. Let  $X$  and  $Y$  be sets.

- (a) Show that  $\mathcal{P}(X \cap Y) = \mathcal{P}(X) \cap \mathcal{P}(Y)$ .
- (b) If  $|X| = n$ ,  $|Y| = m$ , and  $|X \cap Y| = \ell$ , calculate  $|\mathcal{P}(X) \cup \mathcal{P}(Y)|$ .
- (c) With the same numbers as in (b), calculate  $|\mathcal{P}_k(X \cup Y)|$ , where  $0 \leq k \leq |X \cup Y|$ .

**Solution to (a):** ( $\mathcal{P}(X \cap Y) \subseteq \mathcal{P}(X) \cap \mathcal{P}(Y)$ ):

Let  $A \in \mathcal{P}(X \cap Y)$  be arbitrary. Then  $A \subseteq X \cap Y$  by definition. Since  $X \cap Y \subseteq X$  and  $X \cap Y \subseteq Y$ , we have by transitivity of subset inclusion that  $A \subseteq X$  and  $A \subseteq Y$ , i.e.  $A \in \mathcal{P}(X)$  and  $A \in \mathcal{P}(Y)$ , so  $A \in \mathcal{P}(X) \cap \mathcal{P}(Y)$ .

( $\mathcal{P}(X) \cap \mathcal{P}(Y) \subseteq \mathcal{P}(X \cap Y)$ ):

Let  $A \in \mathcal{P}(X) \cap \mathcal{P}(Y)$  be arbitrary. Then  $A \in \mathcal{P}(X)$  and  $A \in \mathcal{P}(Y)$ , thus  $A \subseteq X$  and  $A \subseteq Y$ . Let  $x \in A$  be arbitrary. Then  $x \in X$  and  $x \in Y$ , so  $x \in X \cap Y$ . Hence  $A \subseteq X \cap Y$ , and so  $A \in \mathcal{P}(X \cap Y)$ .

**Solution to (b):** We have

$$\begin{aligned} |\mathcal{P}(X) \cup \mathcal{P}(Y)| &= |\mathcal{P}(X)| + |\mathcal{P}(Y)| - |\mathcal{P}(X) \cap \mathcal{P}(Y)| \\ &= 2^n + 2^m - |\mathcal{P}(X \cap Y)| \\ &= 2^n + 2^m - 2^\ell. \end{aligned}$$

**Solution to (c):** We have

$$\begin{aligned} |\mathcal{P}_k(X \cup Y)| &= \binom{|X \cup Y|}{k} \\ &= \binom{|X| + |Y| - |X \cap Y|}{k} \\ &= \binom{n + m - \ell}{k} \end{aligned}$$

2. Let  $X$  be a set. Define a relation  $\sim$  on  $\mathcal{P}(X)$  by  $A \sim B$  iff  $A = B$  or  $A = X \setminus B$ .
- (a) Show that  $\sim$  is an equivalence relation.
- (b) Let  $x \in X$  be arbitrary. Show that the function

$$f : \mathcal{P}(X)/\sim \rightarrow \mathcal{P}(X)$$

defined by

$$f([A]) = \begin{cases} A & x \in A \\ X \setminus A & x \notin A \end{cases}$$

is well-defined.

**Solution to (a):** Reflexivity: Let  $A \in \mathcal{P}(X)$  be arbitrary. Then  $A = A$ , so  $A \sim A$ .

Symmetry: Suppose  $A \sim B$ . Then either  $A = B$  or  $A = X \setminus B$ . If  $A = B$ , then  $B = A$ , so  $B \sim A$ . If  $A = X \setminus B$ , then  $B = X \setminus (X \setminus B) = X \setminus A$ , so  $B \sim A$ .

Transitivity: Suppose  $A \sim B$  and  $B \sim C$ . Then either  $A = B$  or  $A = X \setminus B$  and either  $B = C$  or  $B = X \setminus C$ , so we have four cases:

- (a)  $A = B$  and  $B = C$ . Then  $A = C$ , so  $A \sim C$ .
- (b)  $A = B$  and  $B = X \setminus C$ . Then  $A = X \setminus C$ , so  $A \sim C$ .
- (c)  $A = X \setminus B$  and  $B = C$ . Then  $A = X \setminus C$ , so  $A \sim C$ .
- (d)  $A = X \setminus B$  and  $B = X \setminus C$ . Then  $C = X \setminus (X \setminus C) = X \setminus B = A$ , so  $A \sim C$ .

**Solution to (b):** Suppose  $[A] = [B]$  for some  $A$  and  $B$ . We need to show that  $f([A]) = f([B])$ . Now,  $[A] = [B]$  iff  $A \sim B$  iff  $A = B$  or  $A = X \setminus B$ . We divide into cases:

- (a) Suppose  $A = B$ , and suppose  $x \in A$ . Then  $x \in B$ , and  $f([A]) = A = B = f([B])$ .
- (b) Suppose  $A = B$ , and suppose  $x \notin A$ . Then  $x \notin B$ , and  $f([A]) = X \setminus A = X \setminus B = f([B])$ .
- (c) Suppose  $A = X \setminus B$ , and suppose  $x \in A$ . Then  $x \notin B$ , and  $f([A]) = A = X \setminus B = f([B])$ .
- (d) Suppose  $A = X \setminus B$ , and suppose  $x \notin A$ . Then  $x \in B$ , and  $f([A]) = X \setminus A = X \setminus (X \setminus B) = B = f([B])$ .

Hence in all cases  $f([A]) = f([B])$ , so  $f$  is well-defined.

3. Let  $(X, d)$  be a metric space. Suppose that every bounded sequence in  $(X, d)$  converges. Show that  $|X| \leq 1$ .  
 [Hint: Consider the sequence  $x, y, x, y, x, y, \dots$ ]

**Note:** There are lots of correct ways to do this; they're all essentially the same, but here are a few examples:

**Solution 1:** We want to show that  $|X| \leq 1$ , i.e. we want to show that either  $X = \emptyset$  or  $|X| = 1$ . So, suppose  $X \neq \emptyset$ . Let  $x, y \in X$  be arbitrary; we want to show  $x = y$ . Consider the sequence  $(x_n)$  in  $(X, d)$ , where

$$x_n = \begin{cases} x & n \text{ odd} \\ y & n \text{ even} \end{cases}$$

This sequence is bounded, since we have

$$d(x, x_n) = \begin{cases} 0 & n \text{ odd} \\ d(x, y) & n \text{ even,} \end{cases}$$

so we have for all  $n \in \mathbb{N}$  that  $d(x, x_n) < d(x, y) + 1$ .

Now, by our assumption on  $(X, d)$ , the sequence  $(x_n)$  must converge to some  $z \in X$ . Let  $\epsilon > 0$  be arbitrary. Then there exists an  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $d(z, x_n) < \frac{\epsilon}{2}$ . Since there are both odd and even numbers larger than  $N$ , we have in particular that  $d(z, x) < \frac{\epsilon}{2}$  and  $d(z, y) < \frac{\epsilon}{2}$ , hence

$$d(x, y) \leq d(x, z) + d(z, y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus  $d(x, y) < \epsilon$  for all  $\epsilon > 0$ . Hence  $d(x, y) = 0$ , so  $x = y$ .

**Solution 2:** Same setup as above. Consider the same (bounded) sequence  $(x_n)$ . By the assumption,  $(x_n)$  is convergent, hence  $(x_n)$  is Cauchy, so there is an  $N \in \mathbb{N}$  such that for all  $n, m \geq N$  we have  $d(x_n, x_m) < \epsilon$ . Choose  $n$  to be odd and  $m$  to be even; then we get  $d(x, y) < \epsilon$ , and we conclude as in Solution 1 that  $x = y$ .

**Solution 3:** By contradiction. Suppose that  $|X| \geq 2$ . Then there exist elements  $x, y \in X$  such that  $x \neq y$ . Thus  $d(x, y) > 0$ . As in Solution 1, we have that  $(x_n)$  is bounded. On the other hand, I claim that  $(x_n)$  is not convergent; this will contradict the assumption that every bounded sequence in  $(X, d)$  is convergent, so we'll be done.

To see that  $(x_n)$  is not convergent, pick a  $z \in X$ . I claim that there is no  $N \in \mathbb{N}$  such that  $d(x_n, z) < \frac{d(x, y)}{2}$  for all  $n \geq N$ . Indeed, suppose there were such an  $N$ ; then choosing an even  $n$  and an odd  $n$ , we would get that

$$d(x, z) < \frac{d(x, y)}{2} \quad \text{and} \quad d(y, z) < \frac{d(x, y)}{2}.$$

Hence we would have

$$d(x, y) \leq d(x, z) + d(z, y) < \frac{d(x, y)}{2} + \frac{d(x, y)}{2} = d(x, y),$$

which is a contradiction.

4. (a) Let  $a, b, c \in \mathbb{N} \setminus \{0\}$ . Show that if  $\gcd(a, b) = 1 = \gcd(a, c)$ , then  $\gcd(a, bc) = 1$ .
- (b) Let  $a, b_1, \dots, b_n \in \mathbb{N} \setminus \{0\}$  be arbitrary. Show that if  $\gcd(a, b_i) = 1$  for all  $i \in \{1, \dots, n\}$ , then  $\gcd(a, b_1 \cdot b_2 \cdots b_n) = 1$ .

**Note:** There are several different ways to solve (a). Here are two of them. The first is extremely slick but not easy to see, whereas the second takes a bit more work but (I think) is more intuitive.

**Solution 1 to (a):** Suppose  $\gcd(a, b) = 1 = \gcd(a, c)$ . Then there exist numbers  $p, q, r, s \in \mathbb{Z}$  such that

$$ap + bq = 1 \quad \text{and} \quad ar + cs = 1.$$

Multiply these two equations together. We get:

$$\begin{aligned} 1 &= (ap + bq)(ar + cs) \\ &= a(arp + pcs + bqr) + bc(qs). \end{aligned}$$

so 1 is an integer linear combination of  $a$  and  $bc$ , hence  $\gcd(a, bc) = 1$ .

**Solution 2 to (a):** We prove the contrapositive. Suppose  $\gcd(a, bc) \neq 1$ . Then  $\gcd(a, bc) > 1$ , so there is a prime  $p \in \mathbb{N}$  such that  $p \mid \gcd(a, bc)$ . Hence  $p \mid a$  and  $p \mid bc$ .

From  $p \mid bc$ , we deduce that either  $p \mid b$  or  $p \mid c$ . In the first case, we have  $p \mid a$  and  $p \mid b$ , so  $p \mid \gcd(a, b)$ , and hence  $\gcd(a, b) \neq 1$ . Likewise in the second case we have that  $p \mid \gcd(a, c)$ , so  $\gcd(a, c) \neq 1$ . Hence either  $\gcd(a, b) \neq 1$  or  $\gcd(a, c) \neq 1$ .

**Solution to (b):** By induction. Let  $P(n)$  be the statement “for any  $a, b_1, \dots, b_n \in \mathbb{N} \setminus \{0\}$ , if  $\gcd(a, b_i) = 1$  for all  $i \in \{1, \dots, n\}$ , then  $\gcd(a, b_1 \cdot b_2 \cdots b_n) = 1$ .” The statement  $P(1)$  is trivial, and we proved  $P(2)$  in part (a).

So, suppose  $P(k)$  is true for some  $k \geq 2$ . We want to prove  $P(k + 1)$ . So, let  $a, b_1, \dots, b_{k+1}$  be arbitrary, and suppose that  $\gcd(a, b_i) = 1$  for all  $i \in \{1, \dots, k + 1\}$ . Then by induction,  $\gcd(a, b_1 \cdot b_2 \cdots b_k) = 1$ . Now,  $b_1 \cdot b_2 \cdots b_{k+1} = (b_1 \cdot b_2 \cdots b_k) \cdot b_{k+1}$ . Since  $\gcd(a, b_1 \cdot b_2 \cdots b_k) = 1$  and  $\gcd(a, b_{k+1}) = 1$ , we conclude from (a) that  $\gcd(a, b_1 \cdot b_2 \cdots b_{k+1}) = 1$ , as desired.