Math 74 Midterm 2 Practice Problems

November 5, 2008

Easier Problems

1. Let X be a set, and let $\mathrm{Rel}(X)$ be the set of relations on X. If $|X| = n$, calculate $|\mathrm{Rel}(X)|$.

2. Find all equivalence relations on the set $X = \{0, 1, 2, 3\}$. For each equivalence relation $\sim$ on X, calculate $|X/{\sim}|$.

3. Let $n, m \in \mathbb{N} \setminus \{0\}$ such that $n \mid m$.

   (a) Show that if $a \equiv b (\text{mod } m)$ then $a \equiv b (\text{mod } n)$. Give an example that shows that the converse is false.

   (b) Show that part (a) implies that the function $f : \mathbb{Z}/{\sim_m} \to \mathbb{Z}/{\sim_n}$ defined by $f([a]_m) = [a]_n$ is well-defined.

   (c) Use this to show that the equation

   $$3x^2 + 3x + 1 \equiv 0 (\text{mod } 59706831)$$

   has no solution.

4. Let X be a set and let d be the discrete metric on X. Show that a sequence (x_n) in X converges if and only if it is eventually constant, i.e. if and only if there exists an $N \in \mathbb{N}$ such that for all $n, m \geq N$ we have $x_n = x_m$.

Medium Problems

5. Recall that we showed that the relation R on $\mathbb{N} \setminus \{0\}$ given by aRb iff $a \mid b$ is a partial order relation, and that we defined $\gcd(a, b)$ to be the greatest lower bound of the set $\{a, b\}$ with respect to this relation, and we defined $\mathrm{lcm}(a, b)$ to be the least upper bound of the set $\{a, b\}$ with respect to this relation. Let $a, b \in \mathbb{N} \setminus \{0\}$ be arbitrary.

(a) Show that a and $b/\gcd(a,b)$ are relatively prime.

(b) Show that if $c, d \in \mathbb{N} \setminus \{0\}$ are relatively prime and $c \mid e$ and $d \mid e$, then $cd \mid e$.

(c) Show that $\operatorname{lcm}(a,b) = \frac{ab}{\gcd(a,b)}$.

6. Let $a, b \in \mathbb{N} \setminus \{0\}$. Here are two possible definitions of the least common multiple of a and b:

Definition 1: The *least common multiple* of a and b is smallest number $c \in \mathbb{N} \setminus \{0\}$ such that $a \mid c$ and $b \mid c$.

Definition 2: The *least common multiple* of a and b is a natural number $e \in \mathbb{N} \setminus \{0\}$ such that:

(a) $a \mid e$ and $b \mid e$

(b) For all $c \in \mathbb{N} \setminus \{0\}$, if $a \mid c$ and $b \mid c$, then $e \mid c$.

Show that these two definitions agree, i.e. they define the same number.

7. Let $a, m \in \mathbb{N} \setminus \{0\}$, and let $b \in \mathbb{Z}$. Show that the equation

$$ax \equiv b (\text{mod } m)$$

has a solution if and only if $\gcd(a,m) \mid b$.

8. Let $E(\mathbb{N})$ be the set of all equivalence relations on $\mathbb{N}$. Is $E(\mathbb{N})$ countable?

Harder Problems

9. Let $(X, \leq_X)$ and $(Y, \leq_Y)$ be partially ordered sets (i.e. X is a set and $\leq_X$ is a partial order on X, and similarly for Y). A function $f : X \to Y$ is called *increasing* if $x_1 \leq_X x_2$ implies $f(x_1) \leq_Y f(x_2)$.

(a) Let $A \subseteq X$, and suppose that A has a least upper bound x in $(X, \leq_X)$ and that $f(A)$ has a least upper bound y in $(Y, \leq_Y)$. Show that $y \leq_Y f(x)$. Give a counterexample that shows that the inequality $f(x) \leq_Y y$ can be false.

(b) Let A and B be sets, and let $f : A \to B$ be any function. Show that the induced function $P(f) : P(A) \to P(B)$ is increasing with respect to the partial orders of inclusion (i.e. $\subseteq$) on $P(A)$ and $P(B)$.

(c) Define a partial order $\leq$ on the set $\{0,1\}$ by $0 \leq 0$, $0 \leq 1$, and $1 \leq 1$. Is the set of increasing functions from $(\mathbb{N}, \leq)$ to $(\{0,1\}, \leq)$ countable?

10. For each number $n \in \mathbb{N}$, let B_n be the number of partitions of the set $A_n = \{0, 1, 2, \ldots, n-1\}$. Equivalently, B_n is the number of equivalence relations on A_n. Show that the numbers B_n satisfy the formula:

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k.$$

Use this formula to calculate B_6.