Math 74 Midterm 1 Solutions

September 24, 2008

1. Let P, Q, and R be propositions. Let A be the proposition

$$P \Rightarrow ((Q \Rightarrow R) \text{ and } (R \Rightarrow Q)).$$

Let B be the proposition

$$(\text{not } P \text{ and } (Q \text{ or } R)) \text{ or } (P \text{ and } (Q \text{ and } R)).$$

Use truth tables to decide whether or not A and B are logically equivalent.

Solution: Write out the truth tables for the two formulae and you'll see that if P is true and Q and R are false, then A is true but B is false, so the formulae are not logically equivalent. Of course, a correct solution would involve actually writing up the truth tables. Please come see me at office hours if you missed this question and don't understand how to solve it!

2. (a) Let $f : \mathbb{Z} \to \mathbb{Z}$ be the function defined by $f(n) = 3n + 2$.

   i. Is f injective? Prove your answer.

   Solution: f is injective: suppose $f(n) = f(m)$ for some $n, m \in \mathbb{Z}$. Then by definition, $3n + 2 = 3m + 2$, so subtracting 2 from both sides, we have $3n = 3m$, and now dividing both sides by 3, we have $n = m$. Hence f is injective.

   ii. Is f surjective? Prove your answer.

   Solution: f is *not* surjective. There is no $n \in \mathbb{Z}$ such that $f(n) = 3$, since if $n \leq 0$ then $f(n) = 3n + 2 \leq 2 < 3$ and if $n \geq 1$ then $f(n) = 3n + 2 \geq 5 > 3$. Hence in either case $f(n) \neq 3$.

(b) Let $f : \mathbb{Z} \to \mathbb{Z}$ be the function defined by $f(n) = \begin{cases} n & n \text{ even}, \\ \frac{n-1}{2} & n \text{ odd}. \end{cases}$

i. Is f injective? Prove your answer.

Solution: f is *not* injective, since $f(0) = 0 = f(1)$.

ii. Is f surjective? Prove your answer.

Solution: f is surjective. Let $n \in \mathbb{Z}$ be arbitrary. We need to find an $m \in \mathbb{Z}$ such that $f(m) = n$. Let $m = 2n + 1$. Then m is odd, so $f(m) = \frac{m-1}{2} = \frac{2n+1-1}{2} = n$, as desired.

3. Let X and Y be sets, let $f : X \to Y$ be a function, and let $P(X)$ and $P(Y)$ denote the power sets of X and Y, respectively. Let $P(f) : P(X) \to P(Y)$ be the function defined by $P(f)(A) = \{f(a) \mid a \in A\}$.

(a) Show that if $P(f)$ is injective, then f is injective. (Hint: think about one-element subsets of X).

Solution 1: Suppose $P(f)$ is injective. We want to show that f is injective, so suppose that $f(x_1) = f(x_2)$ for some $x_1, x_2 \in X$. Then $P(f)(\{x_1\}) = \{f(x_1)\} = \{f(x_2)\} = P(f)(\{x_2\})$, so since $P(f)$ was assumed to be injective, we conclude that $\{x_1\} = \{x_2\}$. Hence $x_1 \in \{x_1\} = \{x_2\}$, but $\{x_2\}$ has only one element, namely x_2, so $x_1 = x_2$, and thus f is injective.

Solution 2: Assume that $P(f)$ is injective. Let $\eta_X : X \to P(X)$ be the function defined by $\eta_X(x) = \{x\}$. We showed in class that η_X is always injective. Hence $P(f) \circ \eta_X$ is injective since the composition of injective functions is injective. But we proved in class that $P(f) \circ \eta_X = \eta_Y \circ f$, so also $\eta_Y \circ f$ is injective. In the homework we showed that if $h \circ k$ is injective, then k is injective, so applying this fact to $\eta_Y \circ f$ tells us that f is injective, as desired. (I certainly didn't expect this proof from anyone, but I wanted to show you it, because it's very pretty).

(b) Show that if f is injective, then $P(f)$ is injective.

Solution: Suppose that f is injective. We want to show that $P(f)$ is injective, so suppose that $P(f)(A) = P(f)(B)$ for some $A, B \in P(X)$. We want to show that this implies that $A = B$. So, let $a \in A$ be arbitrary. We have that $f(a) \in P(f)(A) = P(f)(B) = \{f(b) \mid b \in B\}$, so $f(a) = f(b)$ for some $b \in B$. Since f is injective, we get that $a = b \in B$, i.e. $a \in B$. Hence $A \subseteq B$. Switching the roles of A and B, the exact same argument shows that $B \subseteq A$, so $A = B$, as desired.

4. Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function. We define f to be *eventually increasing* if for each $n \in \mathbb{Z}$ there is an $m \in \mathbb{Z}$ such that $m > n$ and $f(m) > f(n)$.

   (a) Write the above definition using quantifiers.

   Solution: $\forall n \in \mathbb{Z}, \exists m \in \mathbb{Z}, m > n$ and $f(m) > f(n)$.

   (b) Again using quantifiers, write what it means for a function f to *not* be eventually increasing.

   Solution: $\exists n \in \mathbb{Z}, \forall m \in \mathbb{Z}, m \leq n$ or $f(m) \leq f(n)$. Alternatively, I really like the formulation:

   $$\exists n \in \mathbb{Z}, \forall m \in \mathbb{Z}, m > n \Rightarrow f(m) \leq f(n).$$

   (c) Translate your answer from (b) into English.

   Solution: For f not to be eventually increasing means that there is some $n \in \mathbb{Z}$ such that for every $m \in \mathbb{Z}$, either $m \leq n$ or $f(m) \leq f(n)$.

   (d) Define a function $f : \mathbb{Z} \to \mathbb{Z}$ by

   $$f(n) = \begin{cases} 0 & n \text{ even}, \\ 1 & n \text{ odd}. \end{cases}$$

   Prove that f is not eventually increasing.

   Solution: Use part (b). We want to show that we can find an n so that for every $m \in \mathbb{Z}$, either $m \leq n$ or $f(m) \leq f(n)$. Let $n = 1$. Then $f(n) = 1$. Let $m \in \mathbb{Z}$ be arbitrary. Then $f(m) = 0$ or $f(m) = 1$, so in any case $f(m) \leq 1 = f(n)$, which was what we wanted to show.