Math 74 Midterm 1 Practice Problems: Solutions to Selected Problems

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Medium Problems

1. Let X, Y, and Z be sets and let $f : X \to Y$ and $g : Y \to Z$ be functions. Show that $(1_X \times g) \circ \Gamma_f = \Gamma_{g \circ f}$ where $\Gamma_f : X \to X \times Y$ is the graph of f, as defined in class, and $(1_X \times g) : X \times Y \to X \times Z$ is as defined in the homework.

   Note: Recall that the function Γ_f is defined by $\Gamma_f(x) = (x, f(x))$ for all $x \in X$, the function $\Gamma_{g \circ f}$ is defined analogously, and the function $(1_X \times g)$ is defined by $(1_X \times g)((x, y)) = (1_X(x), g(y))$.

   Solution: Let $x \in X$ be arbitrary. We calculate:

$$\begin{aligned} ((1_X \times g) \circ \Gamma_f)(x) &= (1_X \times g)(\Gamma_f(x)) \\ &= (1_X \times g)((x, f(x))) \\ &= (1_X(x), g(f(x))) \\ &= (x, (g \circ f)(x)) \\ &= \Gamma_{g \circ f}(x). \end{aligned}$$

Harder Problems

1. For two sets X and Y, let $\mathrm{Func}(X, Y)$ denote the set of functions from X to Y. If X, Y, and Z are three sets, and $f : X \to Y$ is a function, we get a function $f_* : \mathrm{Func}(Z, X) \to \mathrm{Func}(Z, Y)$ defined by $f_*(g) = f \circ g$ for all $g \in \mathrm{Func}(Z, X)$. Show that f_* is injective if and only if f is injective.

   Correction: For the "only if" direction, we have to assume Z is nonempty.

Solution: ($\Rightarrow$): Suppose $f_* : \mathrm{Func}(Z, X) \to \mathrm{Func}(Z, Y)$ is injective. Let x_1 and x_2 be two elements of X, and suppose $f(x_1) = f(x_2)$. Let $g_1 : Z \to X$ be the function defined by $g_1(z) = x_1$ for all $z \in \mathbb{Z}$ and let $g_2 : Z \to X$ be the function defined by $g_2(z) = x_2$ for all $z \in Z$.

I claim that $f_*(g_1) = f_*(g_2)$. To see this, let $z \in Z$ be arbitrary. Then $(f_*(g_1))(z) = (f \circ g_1)(z) = f(x_1) = f(x_2) = (f \circ g_2)(z) = (f_*(g_2))(z)$. Since we assumed that f_* was injective, it follows that $g_1 = g_2$. Let $z_0 \in Z$ be an element (here we used that $Z \neq \emptyset$!). Then $x_1 = g_1(z_0) = g_2(z_0) = x_2$. Hence f is injective.

($\Leftarrow$): Suppose that f is injective, and suppose $g_1, g_2 : Z \to X$ are two functions such that $f_*(g_1) = f_*(g_2)$, i.e. $(f \circ g_1) = (f \circ g_2)$. We want to show that $g_1 = g_2$. Let $z \in Z$ be arbitrary. Then $f(g_1(z)) = f(g_2(z))$. Since f is injective, it follows that $g_1(z) = g_2(z)$. Since z was arbitrary, it follows that $g_1 = g_2$, as desired.

2. Let X and Y be sets, and let $f : X \to Y$ be a surjective function. Show that there is a subset $A \subseteq X$ such that $f|_A : A \to Y$ is a bijection.

Solution: Since f is surjective, for each $y \in Y$, we can pick an $x_y \in X$ such that $f(x_y) = y$. Having done so, let $A = \{x_y \mid y \in Y\}$. I claim that $f|_A : A \to Y$ is bijective.

($f|_A$ is injective): Let $a_1, a_2 \in A$ be arbitrary, and suppose $f|_A(a_1) = f|_A(a_2)$. By the definition of A, there exist elements $y_1, y_2 \in Y$ such that $a_1 = x_{y_1}$ and $a_2 = x_{y_2}$. Moreover, we have:

$$y_1 = f(x_{y_1}) = f(a_1) = f|_A(a_1) = f|_A(a_2) = f(a_2) = f(x_{y_2}) = y_2.$$

Hence since $y_1 = y_2$ we have $a_1 = x_{y_1} = x_{y_2} = a_2$, as desired.

($f|_A$ is surjective): Let $y \in Y$ be arbitrary. Then $x_y \in A$ and $f|_A(x_y) = f(x_y) = y$, so f is surjective.