Math 74 Midterm 1 Practice Problems

September 17, 2008

Easier Problems

1. Define what it means for a function to be *injective*, and use quantifier negation to explain – both in logical notation and in English – what it means for a function to *not* be injective.

2. Give (with proof) an example of a function $f : \mathbb{Z} \to \mathbb{Z}$ which is injective but not surjective. Give (with proof) an example of a function $g : \mathbb{Z} \to \mathbb{Z}$ which is surjective but not injective.

3. Use the contrapositive to show that if n^2 is an odd number, then n is an odd number.

4. Use proof by contradiction to show that 11 does not divide 36.

Medium Problems

1. Let X, Y, and Z be sets and let $f : X \to Y$ and $g : Y \to Z$ be functions. Show that $(1_X \times g) \circ \Gamma_f = \Gamma_{g \circ f}$ where $\Gamma_f : X \to X \times Y$ is the graph of f, as defined in class, and $(1_X \times g) : X \times Y \to X \times Z$ is as defined in the homework.

2. Let X and Y be sets, and let $f : X \to Y$ be a function (not necessarily bijective). For each subset $A \subseteq Y$, define the *preimage* $f^{-1}(A)$ of A under f by $f^{-1}(A) = \{x \in X \mid f(x) \in A\}$.

   (a) Show that $f^{-1}(Y) = X$.

   (b) Show that if A and B are two subsets of Y, then $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

Harder Problems

1. For two sets X and Y, let $\text{Func}(X, Y)$ denote the set of functions from X to Y. If X, Y, and Z are three sets, and $f : X \to Y$ is a function, we get a function $f_* : \text{Func}(Z, X) \to \text{Func}(Z, Y)$ defined by $f_*(g) = f \circ g$ for all $g \in \text{Func}(Z, X)$. Show that f_* is injective if and only if f is injective.

2. A function $f : \mathbb{R} \to \mathbb{R}$ is called *Lipschitz* if there is a constant $K \in \mathbb{R}$ such that $|f(x) - f(y)| \leq K\,|x - y|$ for all real numbers x and y. Use quantifier negation to explain what it means for a function *not* to be Lipschitz, both in logical notation and in English. Use this to show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $n \mapsto 2^n$ is not Lipschitz.

3. Let X and Y be sets, and let $f : X \to Y$ be a surjective function. Show that there is a subset $A \subseteq X$ such that $f|_A : A \to Y$ is a bijection.