Math 74 Homework 9: Selected Solutions

November 10, 2008

1. Show that each of the following relations is an equivalence relation. In each case, identify the equivalence classes.

   (a) The relation R on $\mathbb{Z}$ given by xRy if $|x| = |y|$.

   (b) The relation R on $\mathbb{Z}$ given by xRy if $2x + y$ is divisible by 3.

   (c) The relation R on $\mathbb{Q} \times \mathbb{Q} \setminus \{(0,0)\}$ given by $(a,b)R(c,d)$ if $ad = bc$. Why did we need to remove $(0,0)$ for this to work?

   Solution to b): Reflexivity: For any $x \in \mathbb{Z}$, $2x + x = 3x$, and $3 \mid 3x$, so xRx.
   Symmetry: Suppose xRy, so $3 \mid (2x+y)$. Since also $3 \mid (3x+3y)$, we have $3 \mid (3x + 3y - 2x - y) = (2y + x)$, so yRx.
   Transitivity: Suppose xRy and yRz. Then $3 \mid (2x + y)$ and $3 \mid (2y + z)$. Hence $3 \mid (2x + 3y + z)$. Since $3 \mid 3y$, we have $3 \mid (2x + 3y + z - 3y) = (2x + z)$, so xRz.

   Now, notice that $0Rx$ iff $3 \mid x$, $1Rx$ iff $3 \mid 2 + x$ iff $3 \mid x - 1$, and $2Rx$ iff $3 \mid 4 + x$ iff $3 \mid x - 2$. So the equivalence classes are the same as the equivalence classes of integers mod 3.

   Solution to c): Reflexivity: For any $(a,b) \in \mathbb{Q} \times \mathbb{Q} \setminus \{(0,0)\}$, we have $ab = ba$, hence $(a,b)R(a,b)$.
   Symmetry: Suppose $(a,b)R(c,d)$. Then $ad = bc$, so $cb = da$, hence $(c,d)R(a,b)$.
   Transitivity: Suppose $(a,b)R(c,d)$ and $(c,d)R(e,f)$. Then $ad = bc$ and $cf = de$. Now, either $c \neq 0$ or $d \neq 0$ by assumption. If $d \neq 0$, multiply the equation $ad = bc$ by f, and we get $adf = bcf = bde$. Dividing by d gives $af = be$, hence $(a,b)R(e,f)$. If $d = 0$, then $c \neq 0$, so the equation $cf = de = 0$ gives that $f = 0$, and the equation $bc = ad = 0$ gives that $b = 0$. Hence

$af = 0 = eb$, so $(a,b)R(e,f)$.

We get a nice description of the equivalence classes as follows: if $b \neq 0$, then $(a,b)R(c,d)$ iff $d \neq 0$ and $a/b = c/d$, hence $[(a,b)] = \{(c,d) \in \mathbb{Q} \times \mathbb{Q} \setminus \{(0,0)\} \mid a/b = c/d\} = \{(d \cdot (a/b), d) \mid d \in \mathbb{Q} \setminus \{0\}\}$. If $b = 0$ then $(a,b)R(c,d)$ iff $d = 0$, i.e. $[(a,b)] = \{(c,0) \mid c \in \mathbb{Q} \setminus \{0\}\}$.

We needed to remove the point $(0,0)$, since otherwise we would have $(a,b)R(0,0)$ for all (a,b), and this would cause transitivity to fail.

2. More examples of relations.

   (a) Let X be a set and let R be the relation on X given by xRy if $x \neq y$. Show that R is symmetric. Show that if $X \neq \emptyset$, then R is not reflexive. Show that if $|X| \geq 2$, then R is not transitive. (Recall $|X| \geq 2$ means there exists an injection $f : A_2 \to X$).

   (b) Give an example of a relation on $\mathbb{Z}$ which is reflexive and symmetric, but not transitive.

   (c) Give an example of a relation on $\mathbb{Z}$ which is symmetric and transitive, but not reflexive.

   (d) Give an example of a relation on $\mathbb{Z}$ which is reflexive and transitive, but not symmetric.

   Solution to a): R is symmetric since $x \neq y$ iff $y \neq x$. If $X \neq \emptyset$, then there exists an $x \in X$, and $x = x$, so $\neg(xRx)$, hence R is not reflexive. If $|X| \geq 2$, then there exist $x, y \in X$ with $x \neq y$, i.e. xRy. Then xRy and yRx, but $\neg(xRx)$, hence R is not transitive.

   Solution to b): We've seen one of these already: Let xRy iff $|x - y| \leq 1$.

   Solution to c): The empty relation has this property. Another nice example: xRy iff x and y are both not equal to 0.

   Solution to d): The relation xRy iff $x \leq y$ has this property.

3. Let X and Y be sets, let R be an equivalence relation on X, and let S be an equivalence relation on Y. Let $\pi_X : X \to X/R$ and $\pi_Y : Y \to Y/S$ be the corresponding natural maps. Suppose that

$f : X \to Y$ is a function such that for all $x_1, x_2 \in X$, if $x_1 R x_2$, then $f(x_1) S f(x_2)$. Show that there is a unique function $g : (X/R) \to (Y/S)$ such that $g \circ \pi_X = \pi_Y \circ f$.

Solution: For an $x \in X$, the equation $(g \circ \pi_X)(x) = (\pi_Y \circ f)(x)$ says that $g([x]_R) = [f(x)]_S$. This totally determines what the function g must be, so uniqueness is clear. We need to show that the function defined by $g([x]_R) = [f(x)]_S$ is well-defined.
So, suppose $[x_1]_R = [x_2]_R$. Then $(x_1)R(x_2)$, so $(f(x_1))S(f(x_2))$ by assumption, and hence $[f(x_1)]_S = [f(x_2)]_S$, i.e.

$$g([x_1]_R) = [f(x_1)]_S = [f(x_2)]_S = g([x_2]_R,$$

as desired.

4. Let $\sim$ be the equivalence relation on $\mathbb{N} \times \mathbb{N}$ given by $(a, b) \sim (c, d)$ iff $a + d = c + b$. Define operations $+, -,$ and $\cdot$ on $\mathbb{N} \times \mathbb{N}/\sim$ by:

   (a) $[(a, b)] + [(c, d)] := [(a + c, b + d)]$,
   (b) $[(a, b)] - [(c, d)] := [(a + d, b + c)]$,
   (c) $[(a, b)] \cdot [(c, d)] := [(ac + bd, ad + bc)]$.

   Show that all of these operations are well-defined (we proved (a) in class).

Solution to c) : This is pretty tricky. We need a fact which I did not state we were allowed to use (hence why this problem wasn't graded): we need to know that for all $n \in \mathbb{N}$, the function $g : \mathbb{N} \to \mathbb{N}$ defined by $g(m) = m + n$ is injective. This actually follows from the axioms for the natural numbers, but we don't know those axioms.

Granted that, here's how you prove this one: suppose $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$, i.e.

$$a + b' = a' + b \quad \text{and} \quad c + d' = c' + d.$$

This gives us the following four equations:

$$\begin{aligned} ac + b'c &= a'c + bc \\ a'd + bd &= ad + b'd \\ ca' + d'a' &= c'a' + da' \\ c'b' + db' &= cb' + d'b'. \end{aligned}$$

Summing gives

$$ac + bd + a'd' + b'c' + (b'c + a'd + a'c + b'd) =$$

$$a'c' + b'd' + ad + bc + (b'c + a'd + a'c + b'd).$$

The injectivity of addition (described above) gives

$$ac + bd + a'd' + b'c' = a'c' + b'd' + ad + bc,$$

hence

$$[(ac + bd, ad + bc)] = [(a'c' + b'd', a'd' + b'c')],$$

as desired.

5. Let $\sim$ be the relation on $\mathbb{N} \times \mathbb{N}$ as in the previous problem, and let $f : (\mathbb{N} \times \mathbb{N}/\sim) \to \mathbb{Z}$ be the function defined by $f([a, b]) = a - b$. Show that:

   (a) $f([a, b] - [c, d]) = f([a, b]) - f([c, d])$, and

   (b) $f([a, b] \cdot [c, d]) = f([a, b]) \cdot f([c, d])$.

   Solution to b): We have

$$\begin{aligned} f([a, b] \cdot [c, d]) &= f([ac + bd, ad + bc]) \\ &= ac + bd - ad - bc \\ &= (a - b)(c - d) \\ &= f([a, b]) \cdot f([c, d]). \end{aligned}$$