Math 74 Homework 9
Due Monday, October 27th

October 20, 2008

1. Let X and Y be sets and let $f : X \to Y$ be a function.

   (a) (Right cancellation). Suppose that f has a right inverse. If Z is another set, $h, k : Y \to Z$ are two functions, and $h \circ f = k \circ f$, show that $h = k$.

   (b) (Left cancellation). Suppose that f has a left inverse. If Z is another set, $h, k : Z \to X$ are two functions, and $f \circ h = f \circ k$, show that $h = k$.

   (c) Let $\sim$ be an equivalence relation on X, let $g : X \to Y$ be another function, and suppose that $f(x_1) = f(x_2)$ and $g(x_1) = g(x_2)$ whenever $x_1 \sim x_2$. Let $\bar{f} : (X/\sim) \to Y$ and $\bar{g} : (X/\sim) \to Y$ be the functions induced by f and g, respectively. Show that $\bar{f} = \bar{g}$ if and only if $f = g$.

2. Show that each of the following relations is an equivalence relation. In each case, identify the equivalence classes.

   (a) The relation R on $\mathbb{Z}$ given by xRy if $|x| = |y|$.

   (b) The relation R on $\mathbb{Z}$ given by xRy if $2x + y$ is divisible by 3.

   (c) The relation R on $\mathbb{Q} \times \mathbb{Q} \setminus \{(0,0)\}$ given by $(a,b)R(c,d)$ if $ad = bc$. Why did we need to remove $(0,0)$ for this to work?

3. More examples of relations.

   (a) Let X be a set and let R be the relation on X given by xRy if $x \neq y$. Show that R is symmetric. Show that if $X \neq \emptyset$, then R is not reflexive. Show that if $|X| \geq 2$, then R is not transitive. (Recall $|X| \geq 2$ means there exists an injection $f : A_2 \to X$).

(b) Give an example of a relation on $\mathbb{Z}$ which is reflexive and symmetric, but not transitive.

(c) Give an example of a relation on $\mathbb{Z}$ which is symmetric and transitive, but not reflexive.

(d) Give an example of a relation on $\mathbb{Z}$ which is reflexive and transitive, but not symmetric.

4. Let $\sim$ be the equivalence relation on $\mathbb{Z}$ given by $|x| = |y|$, and let $\pi : \mathbb{Z} \to \mathbb{Z}/\sim$ be the natural map. Let $f : \mathbb{Z} \to \mathbb{N}$ be the function $f(a) = a^2$. Show that f induces a function $\bar{f} : (\mathbb{Z}/\sim) \to \mathbb{N}$ such that $\bar{f} \circ \pi = f$. Show moreover that $\bar{f}$ is injective.

5. Let X and Y be sets, let R be an equivalence relation on X, and let S be an equivalence relation on Y. Let $\pi_X : X \to X/R$ and $\pi_Y : Y \to Y/S$ be the corresponding natural maps. Suppose that $f : X \to Y$ is a function such that for all $x_1, x_2 \in X$, if $x_1 R x_2$, then $f(x_1) S f(x_2)$. Show that there is a unique function $g : (X/R) \to (Y/S)$ such that $g \circ \pi_X = \pi_Y \circ f$.

6. Let $\sim$ be the equivalence relation on $\mathbb{N} \times \mathbb{N}$ given by $(a, b) \sim (c, d)$ iff $a + d = c + b$. Define operations $+, -$, and $\cdot$ on $\mathbb{N} \times \mathbb{N}/\sim$ by:

   (a) $[(a, b)] + [(c, d)] := [(a + c, b + d)]$,
   (b) $[(a, b)] - [(c, d)] := [(a + d, b + c)]$,
   (c) $[(a, b)] \cdot [(c, d)] := [(ac + bd, ad + bc)]$.

   Show that all of these operations are well-defined (we proved (a) in class).

7. Let $\sim$ be the relation on $\mathbb{N} \times \mathbb{N}$ as in the previous problem, and let $f : (\mathbb{N} \times \mathbb{N}/\sim) \to \mathbb{Z}$ be the function defined by $f([a, b]) = a - b$. Show that:

   (a) $f([a, b] - [c, d]) = f([a, b]) - f([c, d])$, and
   (b) $f([a, b] \cdot [c, d]) = f([a, b]) \cdot f([c, d])$.

   [Note: for those taking 113, these two properties, together with the fact that $f([a, b] + [c, d]) = f([a, b]) + f([c, d])$ proved in class, say that f is a *ring homomorphism* from $\mathbb{N} \times \mathbb{N}/\sim \to \mathbb{Z}$. In fact, f is an isomorphism, since we also showed in class that f is bijective.]