Math 74 Homework 8: Selected Solutions

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1. Let X be a set (not necessarily finite!). Show that if there exists a surjection $f : X \to \text{Func}(X, X)$, then $|X| = 1$.

   Solution: Assume $|X| \neq 1$. We want to show that there is no surjection $f : X \to \text{Func}(X, X)$. We split into two cases: either $X = \emptyset$ or $|X| \geq 2$. If $X = \emptyset$ then $|Func(X, X)| = 1 > 0 = |X|$, so there can be no surjection from X to $\text{Func}(X, X)$ by the pigeonhole principle. Suppose on the other hand that $|X| \geq 2$. Then there exist two elements $x, y \in X$ such that $x \neq y$. Suppose for contradiction that $f : X \to \text{Func}(X, X)$ is a surjection. Let $g : X \to X$ be defined by:

   $$g(z) = \begin{cases} x & f(z)(z) \neq x \\ y & f(z)(z) = x \end{cases}$$

   Then I claim that g is not in the image of f. Indeed, suppose that $f(z) = g$ for some $z \in X$. Then $f(z)(z) = g(z)$. But if $f(z)(z) \neq x$, then $g(z) = x \neq f(z)(z)$. And if $f(z)(z) = x$, then $g(z) = y \neq x = f(z)(z)$. Hence we have a contradiction, so no such z exists, contradicting our assumption that f is surjective.

2. For each $n \in \mathbb{N}$, let X_n be a non-empty countable set. Show that $\bigcup_{n \in \mathbb{N}} X_n$ is countable. (This is an extremely useful fact, often phrased as "a countable union of countable sets is countable.")

   Solution: Let $X = \bigcup_{n \in \mathbb{N}} X_n$. We know that $\mathbb{N} \times \mathbb{N}$ is countable, so to show that X is countable, it suffices to find a surjection $g : \mathbb{N} \times \mathbb{N} \to X$. Since each X_n is countable and non-empty, for each $n \in \mathbb{N}$ there exists a surjection $f_n : \mathbb{N} \to X_n$. Define g by $g(n, m) = f_n(m)$. I claim that g is a surjection. Indeed, if $x \in X$, then $x \in X_n$ for some n, and hence there exists an $m \in \mathbb{N}$ such that $f_n(m) = x$ since f_n is surjective. Thus $x = f_n(m) = g(n, m)$. Hence g is surjective.

Note: The statement is still true if some of the X_n's are allowed to be empty. Here's one way to deal with this: first, if every X_n is empty, then $X = \emptyset$, so X is certainly countable. On the other hand, suppose that not every X_n is empty. Let $A = \{n \in \mathbb{N} \mid X_n \neq \emptyset\}$. Then A is a nonempty subset of $\mathbb{N}$, hence a nonempty countable set, so there exists a surjection $h : \mathbb{N} \to A$. For each $n \in A$, let $f_n : \mathbb{N} \to X_n$ be a surjection (which exists since if $n \in A$ then X_n is nonempty). Define $g : \mathbb{N} \times \mathbb{N} \to X$ by $g(n, m) = f_{h(n)}(m)$. As above, g is a surjection, so X is countable.

3. Let X be a set, and consider the partial order relation $\subseteq$ on $P(X)$. Let $Y \subseteq P(X)$ be a subset of $P(X)$. Show that $\bigcup_{A \in Y} A$ is an upper bound for Y and $\bigcap_{A \in Y} A$ is a lower bound for Y with respect to $\subseteq$.

   [Note: again, $\bigcup_{A \in Y} A$ means $\{x \in X \mid x \in A$ for some $A \in Y\}$, etc.]

   Quasi-Solution: This is sort of trivial, but I've had a lot of questions about this one, so let me at least explain the definitions again.

   Recall that if Z is a set, R is a partial order relation on Z, and $Y \subseteq Z$, then an element $z \in Z$ is an *upper bound* for Y with respect to R if for all $y \in Y$ we have yRz.

   In this context, $Z = P(X)$ and $R = \subseteq$, and we have that for $Y \subseteq Z$, an element $C \in P(X)$ is an upper bound for Y if for all $A \in Y$ we have $A \subseteq C$. So, to show that $\bigcup_{A \in Y} A$ is an upper bound, let $B \in Y$ be arbitrary. We need to show that

$$B \subseteq \bigcup_{A \in Y} A.$$

   So, let $x \in B$ be arbitrary. Then $x \in B$ and $B \in Y$, so $x \in \bigcup_{A \in Y} A$ by the definition of $\bigcup_{A \in Y} A$.

4. Give an example of a relation on $\mathbb{N}$ which is nonempty and:

   (a) Reflexive, but not transitive.

   (b) Transitive, but not reflexive.

   Solution to (a): Let R be the relation on $\mathbb{N}$ given by aRb if $|a - b| \leq 1$. Then R is reflexive, since $|a - a| = 0 \leq 1$ for all $a \in \mathbb{N}$, but R is not reflexive, since $1R2$ and $2R3$ but not $1R3$.

Solution to (b): Many, many examples exist, but the easiest is probably the relation $<$ on $\mathbb{N}$, which is easily seen to have both properties.