Math 74 Homework 8
Due Monday, October 21st

October 12, 2008

1. Let X be a set (not necessarily finite!). Show that if there exists a surjection $f : X \to \text{Func}(X, X)$, then $|X| = 1$.

2. Let I be a set, and suppose that for each $i \in I$ we are given a set X_i. Define
$$\bigcup_{i \in I} X_i := \{x \mid \exists i \in I \text{ such that } x \in X_i\},$$
and similarly, define
$$\bigcap_{i \in I} X_i := \{x \mid x \in X_i \text{ for all } i \in I\}.$$
Let Y be another set. Prove the infinite distributive laws:
   (a) Show that $\left(\bigcup_{i \in I} X_i\right) \cap Y = \bigcup_{i \in I} (X_i \cap Y)$, and
   (b) show that $\left(\bigcap_{i \in I} X_i\right) \cup Y = \bigcap_{i \in I} (X_i \cup Y)$.

3. For each $n \in \mathbb{N}$, let X_n be a countable set. Show that $\bigcup_{n \in \mathbb{N}} X_n$ is countable. (This is an extremely useful fact, often phrased as "a countable union of countable sets is countable.")

4. Let R be the relation on $\mathbb{N} \setminus \{0\}$ given by nRm if n divides m. Show that R is a partial order relation. Let $n, m \in \mathbb{N} \setminus \{0\}$. Describe the least upper bound and greatest lower bound of $\{n, m\}$ with respect to R.

5. Let X be a set, and consider the partial order relation $\subseteq$ on $P(X)$. Let $Y \subseteq P(X)$ be a subset of $P(X)$. Show that $\bigcup_{A \in Y} A$ is an upper bound for Y and $\bigcap_{A \in Y} A$ is a lower bound for Y with respect to $\subseteq$.
[Note: again, $\bigcup_{A \in Y} A$ means $\{x \in X \mid x \in A \text{ for some } A \in Y\}$, etc.]

6. Give an example of a relation on $\mathbb{N}$ which is nonempty and:

   (a) Reflexive, but not transitive.

   (b) Transitive, but not reflexive.