Math 74 Homework 7:
Selected Solutions

October 21, 2008

1. Show that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ for all $n \in \mathbb{N}$.

   Solution: We have that $P(A_n) = \bigcup_{k=0}^{n} P_k(A_n)$, and that $P_k(A_n) \cap P_\ell(A_n) = \emptyset$ for $k \neq \ell$. Hence we have:

$$\begin{aligned} 2^n &= |P(n)| \\ &= \left| \bigcup_{k=0}^{n} P_k(A_n) \right| \\ &= \sum_{k=0}^{n} |P_k(A_n)| \\ &= \sum_{k=0}^{n} \binom{n}{k}. \end{aligned}$$

2. Show that $|P_k(A_n)| = |P_{n-k}(A_n)|$ for all $n, k \in \mathbb{N}$ with $0 \leq k \leq n$ by:

   (a) Computing the numbers on both sides of the equation, and

   (b) Writing down an explicit bijection $f : P_k(A_n) \to P_{n-k}(A_n)$.

   Solution to (a): We have that

$$|P_k(A_n)| = \binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}$$

   and

$$|P_{n-k}(A_n)| = \binom{n}{n-k} = \frac{n!}{(n-k)! \cdot (n-(n-k))!} = \frac{n!}{k! \cdot (n-k)!},$$

   as desired.

Solution to (b): The idea: choosing k elements from an n-element set is the same as choosing $n-k$ elements to leave out. For example, to choose 63 elements from a 64-element set, just pick which one *not* to choose.

To make this rigorous, let

$$\begin{array}{rccc} \phi: & P_k(A_n) & \to & P_{n-k}(A_n) \\ & B & \mapsto & A_n \setminus B. \end{array}$$

Note that this makes sense since if $B \in P_k(A_n)$, then $|A_n \setminus B| = |A_n| - |B| = n-k$. Similarly, let

$$\begin{array}{rccc} \psi: & P_{n-k}(A_n) & \to & P_k(A_n) \\ & C & \mapsto & A_n \setminus C. \end{array}$$

Then ψ and ϕ are inverses. For example, for any $B \in P_k(A_n)$, we have

$$\psi(\phi(B)) = A_n \setminus (A_n \setminus B) = B.$$

3. Let X be a set, and suppose there is an injective function $f : \mathbb{N} \to X$. Prove by contradiction that X is not a finite set.

Solution: Suppose for contradiction that X is finite, i.e. that there exists a bijection $g : X \to A_n$ for some $n \in \mathbb{N}$. Then $g \circ f : \mathbb{N} \to A_n$ is injective, since it's a composition of injective functions. Restricting $g \circ f$ to $A_{n+1} \subset \mathbb{N}$, we get an injective function $h : A_{n+1} \to A_n$. But the pigeonhole principle states that no such function can exist. Hence our assumption was false, and X is not finite.