Math 74 Homework 7
Due Monday, October 13th

October 5, 2008

1. Recall that for X and Y sets and $f : X \to Y$ a function, we defined a function $P(f) : P(X) \to P(Y)$ by $P(f)(A) = \{f(a) \mid a \in A\}$. Let X, Y, and Z be sets and let $f : X \to Y$ and $g : Y \to Z$ be functions.

   (a) Show that $P(1_X) = 1_{P(X)}$.

   (b) Show that $P(g \circ f) = P(g) \circ P(f)$.

   (c) Use (a) and (b) to show that if $f : X \to Y$ is a bijection, then so is $P(f) : P(X) \to P(Y)$.

   [Note: parts (a) and (b) say that P is what is called a *functor*.]

2. Recall that for each set X and natural number k, we defined the set $P_k(X) := \{A \in P(X) \mid |A| = k\}$. Let X and Y be sets and let $f : X \to Y$ be a bijection.

   (a) Show that if $A \in P_k(X)$, then $P(f)(A) \in P_k(Y)$.

   (b) Use (a) and problem 1 to show that $P(f)$ restricts to a bijection from $P_k(X)$ to $P_k(Y)$.

   (c) Using what we proved in class about A_n, show that if $|X| = n$, then $|P_k(X)| = \binom{n}{k}$.

3. Let $\mathbb{Q}$ stand for the rational numbers (these are real numbers which can be written as fractions $\frac{a}{b}$ with $a, b \in \mathbb{Z}$ and $b \neq 0$). Define a function $f : \mathbb{N} \to \mathbb{Q}$ inductively as follows: let $f(0) = 1$, and for $n > 0$, let $f(n) = \frac{6 \cdot f(n-1)+5}{f(n-1)+2}$. Prove that:

   (a) For all $n \in \mathbb{N}$, $f(n) > 0$, and

   (b) For all $n \in \mathbb{N}$, $f(n) < 5$.

4. Show that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ for all $n \in \mathbb{N}$.

5. Show that $|P_k(A_n)| = |P_{n-k}(A_n)|$ for all $n, k \in \mathbb{N}$ with $0 \leq k \leq n$ by:

   (a) Computing the numbers on both sides of the equation, and

   (b) Writing down an explicit bijection $f : P_k(A_n) \to P_{n-k}(A_n)$.

6. Let X be a set, and suppose there is an injective function $f : \mathbb{N} \to X$. Prove by contradiction that X is not a finite set.