Math 74 Homework 6:
Solutions to Selected Problems

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1. By induction on n, show that for a finite set X, if $|X| = n$, then $|P(X)| = 2^n$.

   Solution: Let $Q(n)$ be the statement "If X is a set with $|X| = n$, then $|P(X)| = 2^n$.

   Induction beginning ($Q(0)$): If $|X| = 0$, then $X = \emptyset$, so $P(X) = \{\emptyset\}$, and hence $|P(X)| = 1 = 2^0$.

   Induction Step: Suppose that $Q(k)$ is true for some $k \in \mathbb{N}$, and let X be a set with $|X| = k+1$. Then $X \neq \emptyset$, so there is an element $x \in X$. We have that $|X \setminus \{x\}| = k+1-1 = k$, so $|P(X \setminus \{x\})| = 2^k$ by induction. Choose a bijection $\phi : P(X \setminus \{x\}) \to A_{2^k}$.

   Let $\psi : X \to A_{2^{k+1}}$ be defined by:

   $$\psi(A) = \begin{cases} \phi(A) & x \notin A \\ \phi(A \setminus \{x\}) + 2^k & x \in A \end{cases}$$

   I claim that ψ is a bijection.

   (ψ is a surjection): Let $n \in A_{2^{k+1}}$ be arbitrary. Suppose first that $n \leq 2^k$. Then since ϕ is a bijection, there is a subset $A \subseteq X \setminus \{x\}$ such that $\phi(A) = n$. But then $A \subseteq X$ also, and $x \notin A$, so by definition of ψ we have $\psi(A) = n$. Suppose on the other hand that $2^k < n \leq 2^{k+1}$. Then $1 \leq n - 2^k \leq 2^k$, and again we can find an $A \subseteq X \setminus \{x\}$ such that $\phi(A) = n - 2^k$. Then $A \cup \{x\} \subseteq X$ and we have $\psi(A \cup \{x\}) = 2^k + \phi((A \cup \{x\}) \setminus \{x\}) = 2^k + \phi(A) = 2^k + n - 2^k = n$.

   (ψ is an injection): Suppose $A, B \in P(X)$ such that $\psi(A) = \psi(B)$. Suppose first that $x \notin A$. Then $\psi(A) = \phi(A) \leq 2^k$. Now, if it were the case that $x \in B$, then $\psi(B) = 2^k + \phi(B \setminus \{x\}) >$

$2^k \geq \psi(A)$, contradicting the assumption that $\psi(A) = \psi(B)$. So we must have that $x \notin B$, hence both A and B are also subsets of $X \setminus \{x\}$, and $\phi(A) = \psi(A) = \psi(B) = \phi(B)$. Since ϕ is a bijection, it follows that $A = B$.

Suppose on the other hand that $x \in A$. The same argument as before shows that $x \in B$. We have that $\phi(A \setminus \{x\}) = \psi(A) - 2^k = \psi(B) - 2^k = \phi(B \setminus \{x\})$. Again since ϕ is a bijection, it follows that $A \setminus \{x\} = B \setminus \{x\}$. Hence $A = (A \setminus \{x\}) \cup \{x\} = (B \setminus \{x\}) \cup \{x\} = B$.

Note: There are several other ways to do this problem; probably the easiest one is to show that there is a bijection from $P(X)$ to $\mathrm{Func}(X, A_2)$.

A simpler solution: Again, by induction. $Q(n)$ is the same statement, and we do the induction beginning the same way.

Induction Step: Assume that $Q(k)$ is true for some $k \in \mathbb{N}$, and let X be a set with $|X| = k + 1$. Then $|X| > 0$, so $X \neq \emptyset$, and hence there is an element $x_0 \in X$. Let $A \subseteq P(X)$ be the set $A = \{C \in P(X) \mid x_0 \notin C\}$ and let $B \subseteq P(X)$ be the set $B = \{C \in P(X) \mid x_0 \in C\}$. We easily check that $P(X) = A \cup B$ and that $A \cap B = \emptyset$. Hence $|P(X)| = |A| + |B|$.

Now, the function $f : A \to P(X \setminus \{x_0\})$ defined by $f(C) = C$ is easily seen to be a bijection. Moreover, the function $g : B \to P(X \setminus \{x_0\})$ defined by $g(C) = C \setminus \{x_0\}$ is a bijection, with inverse $h : P(X \setminus \{x_0\}) \to B$ defined by $h(C) = C \cup \{x_0\}$. Now, $|X \setminus \{x_0\}| = k + 1 - 1 = k$, so by induction, $|P(X \setminus \{x_0\})| = 2^k$. Hence $|A| = 2^k = |B|$, and so $|P(X)| = |A| + |B| = 2^k + 2^k = 2^{k+1}$.

2. Define a function $f : \mathbb{N} \setminus \{0\} \to \mathbb{N}$ which sends a natural number n to the largest *odd* natural number which divides n.

   (a) Explain why f is well-defined, i.e. for $n \in \mathbb{N} \setminus \{0\}$ explain:

   - Why there is *some* odd natural number which divides n, and
   - Why there is a *largest* such number.

   (b) Let $q \in \mathbb{N} \setminus \{0\}$. Show that $f(2q) = f(q)$.

   (c) Show that if $a, b \in \mathbb{N}$ with $0 < a < b$ and $f(a) = f(b)$, then a divides b. (Use induction on the maximum value of b, i.e. let $P(n)$ be the previous statement for $b \leq n$).

   (d) Use the pigeonhole principle together with (c) to show that if $X \subseteq A_{2n}$ and $|X| = n + 1$, then X contains two distinct elements a and b such that a divides b.

Solution to (a): Let $n \in \mathbb{N} \setminus \{0\}$ be arbitrary. Certainly 1 divides n, so 1 is an odd natural number which divides n. On the other hand, any odd natural number which divides n must be less than or equal to n by a result we proved in class. Let S be the set of odd numbers which divide n; by what we just said, $S \neq \emptyset$ and $S \subseteq A_n$. Let $T = \{n - s \mid s \in S\}$. Then T is also a nonempty set of natural numbers, so T has a smallest element of the form $n - s_0$ for some $s_0 \in S$. This says $n - s_0 \leq n - s$ for all $s \in S$, hence $s \leq s_0$ for all $s \in S$, so s_0 is a largest element of S

Note on (a): There is a general result in the book that says that nonempty finite sets of natural numbers always have largest elements. This is easy to prove by induction; see the last exercise on Homework 7 a statement which is proved similarly.

Solution to (b): We prove this by showing that $f(q) \leq f(2q)$ and $f(2q) \leq f(q)$. Now, $f(q)$ is the largest odd number dividing q, so $f(q)$ is an odd number which divides $2q$, and hence is less than or equal to the *largest* odd number which divides $2q$, i.e. $f(q) \leq f(2q)$.

On the other hand, let $n = f(2q)$, so n is the largest odd number dividing $2q$. We'll show that n also divides q, and hence $n \leq f(q)$. We have that $2q = mn$ for some $m \in \mathbb{N}$. Now, n is odd. If m were odd also, then mn would be odd, so could not be equal to the even number $2q$. Hence we must have that m is even, i.e. $m = 2p$ for some $p \in \mathbb{N}$. Then $2q = mn = 2pn$, and dividing both sides by 2 gives $q = pn$, so n divides q, as desired.

Solution to (c): Let $P(n)$ be the statement "If $a, b \in \mathbb{N}$ with $0 < a < b \leq n$ and $f(a) = f(b)$, then a divides b." We prove $P(n)$ by induction on n.

Induction Beginning ($P(0)$): In this case, the statement is vacuously true, since there are no numbers $a, b \in \mathbb{N}$ with $0 < a < b \leq 0$.

Induction Step: Suppose that $P(k)$ is true, i.e. suppose that for all $a, b \in \mathbb{N}$ with $0 < a < b \leq k$, if $f(a) = f(b)$, then a divides b. We want to show $P(k+1)$ is true, i.e. the same statement, but with $0 < a < b \leq k + 1$. So we need only to prove the case when $b = k + 1$. Suppose that $0 < a < k + 1$ and $f(a) = f(k + 1)$. We consider two cases:

Case I: Suppose that $k + 1$ is odd. Then $f(a) = f(k + 1) = k + 1 > a$. On the other hand, the definition of $f(a)$ says that

$f(a) = k + 1$ must divide a, which is impossible since $k + 1 > a$. So we have a contradiction, and we must instead be in:

Case II: Suppose that $k+1$ is even. Write $k+1 = 2q$ for some q. We have that $f(a) = f(k + 1) = f(2q) = f(q)$ by part (b) of this problem. Now, if $a = q$, then $k+1 = 2q = 2a$, so a divides $k+1$. If $a \neq q$, then either $a < q$ or $a > q$. If $a < q$, then by induction, we have that a divides q (since $f(a) = f(q)$ and $q \leq k$), so a divides $k + 1 = 2q$ also. It remains to deal with the case that $a > q$.

So, suppose that $a > q$. In this case, we have again by induction that q divides a. If a is odd, then $f(q) = f(a) = a$, so a divides q, contradicting that $q < a$. So we must have that a is even. Hence $\frac{a}{2}$ is a natural number and we have that since $0 < a < k+1 = 2q$, also $0 < \frac{a}{2} < q \leq k$. Moreover, $f(\frac{a}{2}) = f(a) = f(q)$ by part (b), so by induction, we have that $\frac{a}{2}$ divides q. Hence a divides $2q = k + 1$.

Solution to (d): Let $X \subseteq A_{2n}$ and suppose that $|X| = n + 1$. Consider the restriction of the function f to A_{2n}. For any $k \in A_{2n}$, we have that $f(k)$ is an odd number with $1 \leq f(k) \leq 2n - 1$, i.e. if we let $S = \{k \in A_{2n} \mid k \text{ is odd}\}$ then f restricts to a function $\hat{f} : X \to S$. Now, $|S| = n$ (the function $g : A_n \to S$ defined by $g(k) = 2k-1$ gives a bijection from A_n to S), and by assumption $|X| = n+1$. Hence by the pigeonhole principle, $\hat{f}$ is not injective, i.e. there are two elements $a, b \in X$ such that $f(a) = f(b)$ and $a < b$. By (c), it follows that a divides b, as desired.

3. (The general inclusion-exclusion principle) Let $X_1, \ldots, X_n$ be finite sets. For each $I = \{i_1, \ldots, i_k\} \subseteq A_n$, define

$$X_I := X_{i_1} \cap X_{i_2} \cap \ldots \cap X_{i_k}.$$

Prove by induction on n that

$$\left| \bigcup_{i=1}^{n} X_i \right| = \sum_{\emptyset \neq I \subseteq A_n} (-1)^{|I|-1} |X_I| .$$

(The sum on the right is taken over all nonempty subsets of A_n).

Solution: As suggested, we prove the statement by induction on n.

First we note a fact that will be needed; you can prove this by induction if you don't believe it: if $X_1, \ldots, X_n$, and A are all sets, then

$$\left(\bigcup_{i=1}^{n} X_i\right) \cap A = \bigcup_{i=1}^{n} (X_i \cap A).$$

This is just a more general version of the usual rule that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.

Induction Beginning: When $n = 0$, the assertion is that

$$\left|\bigcup_{i=1}^{0} X_i\right| = \sum_{\emptyset \neq I \subseteq \emptyset} (-1)^{|I|-1} |X_I|.$$

Now, the left-hand side of this equation is $|\emptyset| = 0$. The right-hand side of the equation is also 0, since it's an empty sum (there are no non-empty subsets of the empty set).

Induction Step: Suppose that for any n sets $X_1, \ldots X_n$ we have

$$\left|\bigcup_{i=1}^{n} X_i\right| = \sum_{\emptyset \neq I \subseteq A_n} (-1)^{|I|-1} |X_I|.$$

Let $X_1, \ldots X_{n+1}$ be some $n + 1$ sets. We have

$$\begin{aligned}
\left|\bigcup_{i=1}^{n+1} X_i\right| &= \left|\left(\bigcup_{i=1}^{n} X_i\right) \cup X_{n+1}\right| \\
&= \left|\bigcup_{i=1}^{n} X_i\right| + |X_{n+1}| - \left|\left(\bigcup_{i=1}^{n} X_i\right) \cap X_{n+1}\right| \qquad (0.1) \\
&= \left|\bigcup_{i=1}^{n} X_i\right| + |X_{n+1}| - \left|\bigcup_{i=1}^{n} (X_i \cap X_{n+1})\right|,
\end{aligned}$$

where Equation (0.1) follows from the principle of inclusion-exclusion for two sets.

Now, for each $i \in A_n$, let $Y_i = X_i \cap X_{n+1}$, and for $I = \{a_1, \ldots, a_k\} \subseteq A_n$, define $Y_I = Y_{a_1} \cap \cdots \cap Y_{a_k}$. By definition of Y_i, it follows that for each $I \subseteq A_n$, we have $Y_I = X_I \cap X_{n+1} = X_{I \cup \{n+1\}}$. We have:

$$\left|\bigcup_{i=1}^{n+1} X_i\right| = |X_{n+1}| + \left|\bigcup_{i=1}^{n} X_i\right| - \left|\bigcup_{i=1}^{n} Y_i\right|$$

Now, expand both $|\bigcup_{i=1}^{n} X_i|$ and $|\bigcup_{i=1}^{n} Y_i|$ using the induction hypothesis. We get:

$$|X_{n+1}| + \sum_{\emptyset \neq I \subseteq A_n} (-1)^{|I|-1} |X_I| - \sum_{\emptyset \neq I \subseteq A_n} (-1)^{|I|-1} |Y_I| .$$

Using our formula for Y_I and incorporating the negative sign into the power of -1, we get

$$|X_{n+1}| + \sum_{\emptyset \neq I \subseteq A_n} (-1)^{|I|-1} |X_I| + \sum_{\emptyset \neq I \subseteq A_n} (-1)^{|I \cup \{n+1\}|-1} \left|X_{I \cup \{n+1\}}\right| .$$

This formula now clearly simplifies to

$$\sum_{\emptyset \neq I \subseteq A_{n+1}} (-1)^{|I|-1} |X_I| ,$$

as desired.