Math 74 Homework 6
Due Monday, October 6th

September 29, 2008

Remark 0.1. Recall that in class we used the notation A_n for the set $\{1, 2, \dots, n\}$. This notation will be used in this homework.

1. Let X and Y be sets and let $f : X \to Y$ be a function. Recall we defined the *image* of a subset A of X under f, denoted $f(A)$, by

$$f(A) := \{f(a) \mid a \in A\}.$$

   (a) Show that for any $A, B \subseteq X$, $f(A \cup B) = f(A) \cup f(B)$.

   (b) Give a counterexample which shows that it is *not* necessarily the case that $f(A \cap B) = f(A) \cap f(B)$.
   [Note: you might find it interesting to think about *why* this is possible. What condition could you put on f which would guarantee that $f(A \cap B) = f(A) \cap f(B)$ always?]

2. By induction on n, show that for a finite set X, if $|X| = n$, then $|P(X)| = 2^n$.

3. Define a function $f : \mathbb{N} \setminus \{0\} \to \mathbb{N}$ which sends a natural number n to the largest *odd* natural number which divides n.

   (a) Explain why f is well-defined, i.e. for $n \in \mathbb{N} \setminus \{0\}$ explain:
      - Why there is *some* odd natural number which divides n, and
      - Why there is a *largest* such number.

   (b) Let $q \in \mathbb{N} \setminus \{0\}$. Show that $f(2q) = f(q)$.

   (c) Show that if $a, b \in \mathbb{N}$ with $0 < a < b$ and $f(a) = f(b)$, then a divides b. (Use induction on the maximum value of b, i.e. let $P(n)$ be the previous statement for $b \leq n$).

(d) Use the pigeonhole principle together with (c) to show that if $X \subseteq A_{2n}$ and $|X| = n+1$, then X contains two distinct elements a and b such that a divides b.

4. Using the version of the inclusion-exclusion principle proven in class, show that if X, Y, and Z are three finite sets, then:

$$|X \cup Y \cup Z| = |X|+|Y|+|Z|-|X \cap Y|-|X \cap Z|-|Y \cap Z|+|X \cap Y \cap Z|\,.$$

(Note: there is a proof of this in the book; you should try to prove it without looking at that proof, since it's a good warm-up for the next exercise.)

5. (The general inclusion-exclusion principle) Let $X_1, \ldots, X_n$ be finite sets. For each $I = \{i_1, \ldots, i_k\} \subseteq A_n$, define

$$X_I := X_{i_1} \cap X_{i_2} \cap \ldots \cap X_{i_k}.$$

Prove by induction on n that

$$\left| \bigcup_{i=1}^{n} X_i \right| = \sum_{\emptyset \neq I \subseteq A_n} (-1)^{|I|-1} \, |X_I|\,.$$

(The sum on the right is taken over all nonempty subsets of A_n).

[Note: if you find this formula confusing, try to write out what it means when $n = 3$. You should get the same formula as in the previous problem.]

6. How many natural numbers between 1 and 999 have no repeated digits? (Here I count the number 101, for example, as having the repeated digit 1.)