

Math 74 Homework 4 Selected Solutions

September 22, 2008

1. Which of the following functions are injective? Which are surjective? (You have to prove your answers.)
 - (a) $f : \mathbb{Z} \rightarrow \mathbb{Z}, f(n) = n^3$.
 - (b) $g : \mathbb{R} \rightarrow \mathbb{R}, g(n) = n^3$.
 - (c) $h : \mathbb{Z} \rightarrow \mathbb{N}, h(n) = |n|$. (Here \mathbb{N} is the set $\mathbb{N} = \{n \in \mathbb{Z} \mid n \geq 0\}$.)

Solution to a): The function f is injective but not surjective. Proof:

(f is injective): Suppose that $f(n) = f(m)$ for some $n, m \in \mathbb{N}$. Then by definition, $n^3 = m^3$. If $n = 0$, then $n^3 = 0$, so $m^3 = 0$ and hence also $m = 0$ (by the cancellation law for multiplication), so $n = m$. If $n \neq 0$, then we can divide both sides of the equation $n^3 = m^3$ by n^3 to see that $(m/n)^3 = 1$, so it suffices to show that if a is a rational number and $a^3 = 1$, then $a = 1$, since this will show that $m/n = 1$ and hence that $m = n$.

So, to show that if $a^3 = 1$, then $a = 1$, we use the contrapositive. Suppose $a \neq 1$. Then either $a < 1$ or $a > 1$. From the rules for inequalities and multiplication, we see that cubing “preserves inequalities,” so if $a < 1$ then $a^3 < 1^3 = 1$ and similarly if $a > 1$ then $a^3 > 1^3 = 1$, hence in either case $a^3 \neq 1$, as desired.

(f is not surjective): I will show that there is no $n \in \mathbb{Z}$ such that $n^3 = 2$. If $n \leq 1$, then $n^3 \leq 1^3 = 1 < 2$, so $n^3 \neq 2$. And if $n \geq 2$ then $n^3 \geq 8 > 2$, so $n^3 \neq 2$. Hence there is no $n \in \mathbb{Z}$ with $n^3 = 2$, so f is not surjective.

2. Let X, Y , and Z be sets, and let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be functions. Show that:

- (a) If f and g are injective, then $g \circ f$ is injective.
- (b) If f and g are surjective, then $g \circ f$ is surjective.
- (c) If $g \circ f$ is surjective, then g is surjective.
- (d) If $g \circ f$ is injective, then f is injective.

Note: There was a typo in part d), which I didn't catch until now.

The problem claimed that if $g \circ f$ is injective, then g is injective, which is **false** (consider, for example, the case when $X = \{0\} = Z$, $Y = \mathbb{Z}$, $f(0) = 0$, and $g(n) = 0$ for all $n \in \mathbb{Z}$.) I'm surprised no one emailed me about this. It's traditional in mathematics, by the way, that when an exercise has an error, spotting the error becomes part of the exercise. Hope you found it!

Solution to b): Suppose f and g are surjective. Let $z \in Z$ be arbitrary. Then there exists a $y \in Y$ such that $g(y) = z$, and there exists an $x \in X$ such that $f(x) = y$. Hence $(g \circ f)(x) = g(f(x)) = g(y) = z$, so $g \circ f$ is surjective.

Solution to c): Suppose $g \circ f$ is surjective. Let $z \in Z$ be arbitrary. Then there exists an $x \in X$ such that $(g \circ f)(x) = z$, i.e. $g(f(x)) = z$. So $y = f(x)$ is an element of Y such that $g(y) = z$, hence g is surjective.

Solution to d): Suppose that $g \circ f$ is injective. Suppose $f(x_1) = f(x_2)$ for some $x_1, x_2 \in X$. Then $g(f(x_1)) = g(f(x_2))$, i.e. $(g \circ f)(x_1) = (g \circ f)(x_2)$. Since $g \circ f$ is injective, it follows that $x_1 = x_2$, and so f is injective.