Math 74 Homework 15: Selected Solutions

December 9, 2008

1. If (X, d) is a metric space, a subset $Y \subseteq X$ is called *dense* if $\bar{Y} = X$.

   (a) Show that $\mathbb{Q}$ is dense in $\mathbb{R}$ (with the Euclidean metric).

   (b) Show that $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$.

   Solution to (a): By HW 14, Problem 3(c), $\bar{\mathbb{Q}}$ is the set of limit points of $\mathbb{Q}$. So we need to show that every real number is a limit point of $\mathbb{Q}$. By HW 14, Problem 2, this is the same as showing that for each real number $r \in \mathbb{R}$, there is a sequence of rational numbers (q_n) which converges to r. We already know this is true.

   Solution to (b): As in (a), we want to show that every element of $\mathbb{R}$ is a limit point of $\mathbb{R} \setminus \mathbb{Q}$. Since any element of $\mathbb{R} \setminus \mathbb{Q}$ is obviously a limit point of $\mathbb{R} \setminus \mathbb{Q}$, it suffices to show that every element of $\mathbb{Q}$ is a limit point of $\mathbb{R} \setminus \mathbb{Q}$. Let $q \in \mathbb{Q}$ be an arbitrary rational number. I claim that for each $n \in \mathbb{N} \setminus \{0\}$, the number $a_n := q + \frac{\sqrt{2}}{n}$ is irrational (i.e. is an element of $\mathbb{R} \setminus \mathbb{Q}$). To see this, suppose that a_n is rational. Then $n \cdot (a_n - q)$ is also rational. But $n \cdot (a_n - q) = \sqrt{2}$, which is irrational, as we have proven in class.

   Now, I claim that $q = \lim_n a_n$. Indeed, we have that $|q - a_n| = \frac{\sqrt{2}}{n}$. Let $\epsilon > 0$ be arbitrary, and choose and $N \in \mathbb{N} \setminus \{0\}$ such that $N > \frac{\sqrt{2}}{\epsilon}$. Then for $n \geq N$, we have $n > \frac{\sqrt{2}}{\epsilon}$, hence $\epsilon > \frac{\sqrt{2}}{n}$. Thus for all $n \geq N$, we have that $|q - a_n| = \frac{\sqrt{2}}{n} < \epsilon$, as desired.

2. Let X be any set, and let d be the discrete metric. Find all open sets of (X, d).

   Solution: Every subset of (X, d) is open: let $A \subseteq X$ be an arbitrary subset. Then for each $a \in A$, we have that $B(a, \frac{1}{2}) = \{a\} \subseteq A$.

3. A metric space (X, d) is called *connected* if the only subsets of X which are both open and closed are X and $\emptyset$.

   (a) Give (with proof) an example of a metric space which is connected, and an example of a metric space which is *not* connected.

   (b) Suppose (X, d) and (Y, ρ) are two metric spaces, X is connected and $f : X \to Y$ is a surjective continuous function. Show that Y is connected.

   Solution to (a): An easy example of a connected metric space: let X be a one-element set and let d be the only possible metric on X. Then (X, d) is connected since every subset of X is either X or $\emptyset$, so the "open and closed" portion of the criterion need not even be checked.

   An easy example of a metric space which is not connected: let X be any set with $|X| \geq 2$, and let d be the discrete metric on X. By problem 2, every subset of X is both open and closed. Hence, since X has subsets not equal to X or $\emptyset$, X is not connected. [Note: the case $|X| = 2$ gives a very visual example of "not being connected".]

   Solution to (b): Let $U \subseteq Y$ be any subset which is both open and closed. Then $f^{-1}(U)$ is also both open and closed, hence either $f^{-1}(U) = X$ or $f^{-1}(U) = \emptyset$, since X is connected. Now, if $f^{-1}(U) = X$, then $f(X) \subseteq U$. Since f is surjective by assumption, $f(X) = Y$, so $U = Y$. If $f^{-1}(U) = \emptyset$, then $f(X) \cap U = \emptyset$. Again, since $f(X) = Y$, we conclude that $U = \emptyset$. Hence Y is connected.

   Note: It is also true that $\mathbb{R}$ is connected in the Euclidean metric, and that, more generally, $\mathbb{R}^n$ is connected in the Euclidean metric. The proofs are rather more difficult; typically one needs some trick to prove this for $\mathbb{R}$, and then one reduces the case of $\mathbb{R}^n$ to the case of $\mathbb{R}$.

4. Let $c \in \mathbb{R}$ be arbitrary. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(a) = c \cdot a$ is continuous.

   Solution: Let $x \in \mathbb{R}$ and $\epsilon > 0$ be arbitrary. We want to show that there is a $\delta > 0$ such that whenever $|x - y| < \delta$ we get $|cx - cy| < \epsilon$. If $c = 0$, then this last inequality just says $0 < \epsilon$, which is always true, so any δ will work. Assume now that $c \neq 0$. Let $\delta = \frac{\epsilon}{|c|}$. Then if $|x-y| < \delta = \frac{\epsilon}{|c|}$, we have that $|cx - cy| = |c|\,|x-y| < |c|\,\frac{\epsilon}{|c|} = \epsilon$. Hence f is continuous.

5. Bonus problem: Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(0) = 0$, $f(1) = 1$, and for all $a, b \in \mathbb{R}$, $f(ab) = f(a)f(b)$ and $f(a+b) = f(a) + f(b)$. Show that f is the identity function [Hint: first show that $f(q) = q$ for all $q \in \mathbb{Q}$ and that f is increasing. You may assume that square roots of positive real numbers exist.]

   Solution: Note first that since

$$0 = f(0) = f(a + (-a)) = f(a) + f(-a),$$

   we have that $f(-a) = -f(a)$ for all $a \in \mathbb{R}$.

   I claim first that $f(n) = n$ for all $n \in \mathbb{N}$. The cases $n = 0, 1$ are given. Suppose by induction that $f(k) = k$ for some $k \in \mathbb{N}$, $k \geq 1$. Then $f(k+1) = f(k) + f(1) = k + 1$, hence by induction $f(n) = n$ for all $n \in \mathbb{N}$. It follows immediately from the fact that $f(-a) = -f(a)$ for all $a \in \mathbb{R}$ that $f(z) = z$ for all $z \in \mathbb{Z}$.

   Now, let $q \in \mathbb{Q}$ be arbitrary. Then $q = \frac{a}{b}$ for some $a \in \mathbb{Z}$, $b \in \mathbb{Z} \setminus \{0\}$. Then we have that

$$a = f(a) = f(bq) = f(b)f(q) = b \cdot f(q),$$

   hence $f(q) = \frac{a}{b} = q$. Thus $f(q) = q$ for all $q \in \mathbb{Q}$.

   Next we show that f is increasing. To show this, we first note that if $a \geq 0$, then $f(a) \geq 0$. To see this, let $b = \sqrt{a}$. Then $a = b^2$, so $f(a) = f(b^2) = (f(b))^2 \geq 0$. Now, let a and b be any two real numbers. Suppose that $a \geq b$. Then $a - b \geq 0$, hence by the above, $f(a-b) \geq 0$. Now, $f(a-b) = f(a) + f(-b) = f(a) - f(b)$, so $f(a-b) \geq 0$ implies that $f(a) \geq f(b)$. Thus f is increasing.

   Now, let $r \in \mathbb{R}$ be any real number. Let (q_n) be an increasing sequence of rational numbers such that $\lim_n q_n = r$, and let (q'_n) be a decreasing sequence of rational numbers such that $\lim_n q'_n = r$. Since f is increasing, we have that

$$q_n = f(q_n) \leq f(r) \leq f(q'_n) = q'_n$$

for all n. Hence

$$r = \lim_n q_n \leq f(r) \leq \lim_n q'_n = r.$$

Thus $f(r) = r$.