

Math 74 Homework 15  
Due Monday, December 8th

December 2, 2008

1. If  $(X, d)$  is a metric space, a subset  $Y \subseteq X$  is called *dense* if  $\bar{Y} = X$ .
  - (a) Show that  $\mathbb{Q}$  is dense in  $\mathbb{R}$  (with the Euclidean metric).
  - (b) Show that  $\mathbb{R} \setminus \mathbb{Q}$  is dense in  $\mathbb{R}$ .
2. Let  $X$  be any set, and let  $d$  be the discrete metric. Find all open sets of  $(X, d)$ .
3. A metric space  $(X, d)$  is called *connected* if the only subsets of  $X$  which are both open and closed are  $X$  and  $\emptyset$ .
  - (a) Give (with proof) an example of a metric space which is connected, and an example of a metric space which is *not* connected.
  - (b) Suppose  $(X, d)$  and  $(Y, \rho)$  are two metric spaces,  $X$  is connected and  $f : X \rightarrow Y$  is a surjective continuous function. Show that  $Y$  is connected.
4. Let  $c \in \mathbb{R}$  be arbitrary. Show that the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(a) = c \cdot a$  is continuous.
5. Bonus problem: Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function such that  $f(0) = 0$ ,  $f(1) = 1$ , and for all  $a, b \in \mathbb{R}$ ,  $f(ab) = f(a)f(b)$  and  $f(a + b) = f(a) + f(b)$ . Show that  $f$  is the identity function [Hint: first show that  $f(q) = q$  for all  $q \in \mathbb{Q}$  and that  $f$  is increasing. You may assume that square roots of positive real numbers exist.]