Math 74 Homework 14: Selected Solutions

December 2, 2008

1. (a) Show that $\mathbb{R}^2 (= \mathbb{R} \times \mathbb{R})$ is Cauchy complete with respect to the Euclidean metric.
   (b) Show that the set $A = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq \sqrt{x^2 + y^2} \leq 2\}$ is Cauchy complete with respect to the Euclidean metric.

Solution to (a): Let $((x_n, y_n))$ be a Cauchy sequence in $\mathbb{R}^2$. We want to show that this sequence converges. Recall that we proved that $((x_n, y_n))$ converges in $\mathbb{R}^2$ with the Euclidean metric if and only if (x_n) and (y_n) converge in $\mathbb{R}$ with the Euclidean metric. We know now that (x_n) and (y_n) converge in $\mathbb{R}$ if and only if they are Cauchy. So it suffices to show that if $((x_n, y_n))$ is Cauchy, then (x_n) and (y_n) are Cauchy. Now, let $\epsilon > 0$ be arbitrary. Then there exists an $N \in \mathbb{N} \setminus \{0\}$ such that for all $n, m \geq N$, $d((x_n, y_n), (x_m, y_m)) < \epsilon$, i.e.

$$\sqrt{(x_n - x_m)^2 + (y_n - y_m)^2} < \epsilon,$$

hence

$$(x_n - x_m)^2 + (y_n - y_m)^2 < \epsilon^2,$$

and since both $(x_n - x_m)^2$ and $(y_n - y_m)^2$ are positive, we get

$$(x_n - x_m)^2 < \epsilon^2 \qquad \text{and} \qquad (y_n - y_m)^2 < \epsilon^2,$$

hence $|x_n - x_m| < \epsilon$ and $|y_n - y_m| < \epsilon$, as desired.

Solution to (b): Since $\mathbb{R}^2$ is Cauchy complete by (a), it suffices to show that A is closed. Now, let $C = \{(x, y) \in \mathbb{R}^2 \mid \sqrt{x^2 + y^2} \geq 1\}$ and let $D = \{(x, y) \in \mathbb{R}^2 \mid \sqrt{x^2 + y^2} \leq 2\}$, so that $A = C \cap D$. Now, D is closed because D is the closed ball $\bar{B}(0, 2)$, and C is closed because C is the complement of the open ball $B(0, 1)$. Hence $A = C \cap D$ is closed.

2. Let (X, d) be a metric space and let $Y \subseteq X$ be a subset. Show that $x \in X$ is a limit point of Y if and only if there exists a sequence (y_n) of elements of Y such that (y_n) converges to x.

   Solution: By definition, $x \in X$ is a limit point of Y if and only if for every $r \in \mathbb{R}$, $r > 0$, we have $B(x, r) \cap Y \neq \emptyset$. Suppose x is a limit point of Y. For each $n \in \mathbb{N} \setminus \{0\}$, the set $B(x, 1/n) \cap Y$ is not empty, so we can choose a $y_n \in B(x, 1/n) \cap Y$. Then the sequence (y_n) in Y clearly converges to x.

   Suppose on the other hand that there is a sequence (y_n) in Y which converges to x. Let $r \in \mathbb{R}$, $r > 0$ be arbitrary. Then there exists an $N \in \mathbb{N}\setminus\{0\}$ such that for all $n \geq N$ we have $d(y_n, x) < r$. Hence in particular $y_N \in B(x, r)$, so $B(x, r) \cap Y \neq \emptyset$. Hence x is a limit point of Y.

3. Let (X, d) be a metric space and let $Y \subseteq X$ be a subset. Define the *closure* $\bar{Y}$ of Y to be the intersection of all closed subsets of X which contain Y.

   (a) Show that $\bar{Y}$ is the *smallest* closed subset of X containing Y, i.e. show that $\bar{Y}$ is closed, that $Y \subseteq \bar{Y}$, and that if $Z \subseteq X$ is any closed subset of X such that $Y \subseteq Z$, then $\bar{Y} \subseteq Z$.

   (b) Show that the property in (a) totally determines $\bar{Y}$, i.e. show that if $C \subseteq X$ is a closed set such that $Y \subseteq C$ and such that C is contained in any other closed Z which contains Y, then $C = \bar{Y}$.

   (c) Show that $\bar{Y} = \{x \in X \mid x \text{ is a limit point of } Y\}$.

   Solution to (a): $\bar{Y}$ is closed in X since it is the intersection of closed sets. We have $Y \subseteq \bar{Y}$ since obviously Y is contained in any closed set which contains Y, and hence is contained in the intersection of all such sets. Suppose $Z \subseteq X$ is a closed subset of X such that $Y \subseteq Z$. Then Z is one of the elements of the intersection which defines $\bar{Y}$, hence $\bar{Y} \subseteq Z$.

   Solution to (b): Suppose C is as given. By the defining property of C and the fact that $\bar{Y}$ is closed and $Y \subseteq \bar{Y}$, we have that $C \subseteq \bar{Y}$. Since C is closed and $Y \subseteq C$, we have by (a) that $\bar{Y} \subseteq C$, hence $C = \bar{Y}$.

   Solution to (c): Let $A = \{x \in X \mid x \text{ is a limit point of } Y\}$. Let $x \in A$ be arbitrary. Then x is a limit point of Y. For each

$r \in \mathbb{R}, r > 0$, we have $B(x,r) \cap \bar{Y} \supseteq B(x,r) \cap Y \neq \emptyset$, hence x is a limit point of $\bar{Y}$. Since $\bar{Y}$ is closed, we have $x \in \bar{Y}$. Thus $A \subseteq \bar{Y}$.

Now, $Y \subseteq A$, so to show that $\bar{Y} \subseteq A$, it suffices to show that A is closed. To see this, let $x \in X$ be a limit point of A. Then for all $r \in \mathbb{R}$, $r > 0$, we have $B(x,r) \cap A \neq \emptyset$. Let $a \in B(x,r) \cap A$ be arbitrary. Since $B(x,r)$ is open, there is an $r' \in \mathbb{R}$, $r' > 0$ such that $B(a,r') \subseteq B(x,r)$. Now, $B(x,r) \cap Y \supseteq B(a,r') \cap Y \neq \emptyset$, so $B(x,r) \cap Y \neq \emptyset$. Hence x is a limit point of Y, and so $x \in A$. Hence A contains all its limit points, so A is closed.

4. As in the previous exercise, let (X,d) be a metric space and let $Y \subseteq X$ be a subset. Define the *interior* Y° of Y to be the largest open subset of Y. Show that Y° exists and is unique, give a nice description of Y°, and relate Y° to limit points in X.

   Outline: To show that Y° exists, show that it is equal to the union of all open sets of X which are contained in Y (this is also the desired "nice description"). In terms of limit points, Y° is the set of all points of Y which are not limit points of $X \setminus Y$. Another way of phrasing this is to say that $Y^\circ = X \setminus \overline{(X \setminus Y)}$.

5. Let (X,d) be a metric space, and let Y and Z be two nonempty subsets of X. Define the *distance from Y to Z*, $\operatorname{dist}(Y,Z)$, to be the greatest lower bound of set $\{d(y,z) \mid y \in Y, z \in Z\}$. Explain why $\operatorname{dist}(Y,Z)$ exists. For each of the following statements, either prove the statement, or give a counterexample (very convincing pictures are acceptable):

   (a) For any three nonempty subsets $W, Y, Z \subseteq X$,

   $$\operatorname{dist}(Y,Z) \leq \operatorname{dist}(Y,W) + \operatorname{dist}(W,Z).$$

   (b) If $\operatorname{dist}(Y,Z) = 0$ then $Y \cap Z \neq \emptyset$.

   (c) If Y and Z are closed subsets of X and $\operatorname{dist}(Y,Z) = 0$ then $Y \cap Z \neq \emptyset$.

   Solution: All three statements are false. To see that (a) is false, consider the subsets of $X = \mathbb{R}^2$ given by $Y = B(0,1)$, $W = B(2,1)$, $Z = B(4,1)$ (draw a picture if you want this to make any sense!). Then $\operatorname{dist}(Y,Z) = 2$ but $\operatorname{dist}(Y,W) = 0 = \operatorname{dist}(W,Z)$. The sets

Y and W also provide a counterexample to (b).

Seeing that (c) is false requires more work. There's a nice geometric solution which I would be happy to present at office hours, but which I don't have the tools to draw here. Here's another neat solution: let $X = \mathbb{R}$, let $Y = \mathbb{N} \subseteq \mathbb{R}$, and let $Z \subseteq \mathbb{R}$ be the set $Z = \{n + \frac{1}{n} \mid n \in \mathbb{N}, n \geq 2\}$. Then clearly $Y \cap Z = \emptyset$ since $n + \frac{1}{n}$ can't be an integer if $n \geq 2$. On the other hand, $\lim_{n\to\infty} d(n, n + \frac{1}{n}) = 0$, so that we must have $\mathrm{dist}(Y, Z) = 0$. To see that Y and Z are closed, it suffices to note that any convergent sequence of elements of Y must be eventually constant, and similarly for Z, so Y and Z contain all of their limit points.