

Math 74 Homework 14
Due Monday, December 1st

November 24, 2008

1. (a) Show that $\mathbb{R}^2 (= \mathbb{R} \times \mathbb{R})$ is Cauchy complete with respect to the Euclidean metric.
(b) Show that the set $\{(x, y) \in \mathbb{R}^2 \mid 1 \leq \sqrt{x^2 + y^2} \leq 2\}$ is Cauchy complete with respect to the Euclidean metric.
2. Let (X, d) be a metric space and let $Y \subseteq X$ be a subset. Show that $x \in X$ is a limit point of Y if and only if there exists a sequence (y_n) of elements of Y such that (y_n) converges to x .
3. Let (X, d) be a metric space and let $Y \subseteq X$ be a subset. Define the *closure* \bar{Y} of Y to be the intersection of all closed subsets of X which contain Y .
 - (a) Show that \bar{Y} is the *smallest* closed subset of X containing Y , i.e. show that \bar{Y} is closed, that $Y \subseteq \bar{Y}$, and that if $Z \subseteq X$ is any closed subset of X such that $Y \subseteq Z$, then $\bar{Y} \subseteq Z$.
 - (b) Show that the property in (a) totally determines \bar{Y} , i.e. show that if $C \subseteq X$ is a closed set such that $Y \subseteq C$ and such that C is contained in any other closed Z which contains Y , then $C = \bar{Y}$.
 - (c) Show that $\bar{Y} = \{x \in X \mid x \text{ is a limit point of } Y\}$.
4. As in the previous exercise, let (X, d) be a metric space and let $Y \subseteq X$ be a subset. Define the *interior* Y° of Y to be the largest open subset of Y . Show that Y° exists and is unique, give a nice description of Y° , and relate Y° to limit points in X .
5. Let (X, d) be a metric space, and let Y and Z be two nonempty subsets of X . Define the *distance from Y to Z* , $\text{dist}(Y, Z)$, to be the greatest

lower bound of set $\{d(y, z) \mid y \in Y, z \in Z\}$. Explain why $\text{dist}(Y, Z)$ exists. For each of the following statements, either prove the statement, or give a counterexample (very convincing pictures are acceptable):

(a) For any three nonempty subsets $W, Y, Z \subseteq X$,

$$\text{dist}(Y, Z) \leq \text{dist}(Y, W) + \text{dist}(W, Z).$$

(b) If $\text{dist}(Y, Z) = 0$ then $Y \cap Z \neq \emptyset$.

(c) If Y and Z are closed subsets of X and $\text{dist}(Y, Z) = 0$ then $Y \cap Z \neq \emptyset$.