

## Math 74 Homework 13: Selected Solutions

December 2, 2008

1. Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces. Suppose that  $f : X \rightarrow Y$  is a function such that  $\rho(f(x_1), f(x_2)) \leq d(x_1, x_2)$  for all  $x_1, x_2 \in X$ . Show that if  $(x_n)$  is a Cauchy sequence in  $X$ , then  $(f(x_n))$  is a Cauchy sequence in  $Y$ .

**Solution:** Let  $\epsilon > 0$  be arbitrary. Since  $(x_n)$  is Cauchy, there exists an  $N \in \mathbb{N} \setminus \{0\}$  such that for all  $n, m \geq N$ ,  $d(x_n, x_m) < \epsilon$ . Hence by our assumption, for all  $n, m \geq N$ ,  $\rho(f(x_n), f(x_m)) < \epsilon$ . Hence  $(f(x_n))$  is Cauchy.

2. Let  $(X, d)$  be a metric space, and let  $(a_n)$  and  $(b_n)$  be convergent sequences in  $X$  such that  $\lim_{n \rightarrow \infty} a_n \neq \lim_{n \rightarrow \infty} b_n$ . Show that there exists a  $\delta \in \mathbb{R}$ ,  $\delta > 0$ , and an  $N \in \mathbb{N} \setminus \{0\}$  such that for all  $n \geq N$ ,  $d(a_n, b_n) > \delta$ .

Let  $a = \lim_{n \rightarrow \infty} a_n$  and let  $b = \lim_{n \rightarrow \infty} b_n$ . By assumption,  $a \neq b$ . Without loss of generality,  $a < b$ , i.e.  $0 < a - b$ . Let  $\delta = \frac{a-b}{3}$ . Since  $a$  is the limit of  $(a_n)$  and  $b$  is the limit of  $(b_n)$ , there exists an  $N \in \mathbb{N} \setminus \{0\}$  such that for all  $n \geq N$  we have  $d(a_n, a) < \delta$  and  $d(b_n, b) < \delta$ . Hence we have

$$d(a, b) \leq d(a, a_n) + d(a_n, b_n) + d(b_n, b) < 2\delta + d(a_n, b_n),$$

so

$$d(a_n, b_n) > d(a, b) - 2\delta = d(a, b) - \frac{2}{3} \cdot d(a, b) = \frac{d(a, b)}{3} = \delta,$$

as desired.

3. Let  $(X, d)$  be a metric space, and let  $C$  be the set of all Cauchy sequences in  $(X, d)$ . Define an equivalence relation  $\sim$  on  $C$  by  $(x_n) \sim$

$(y_n)$  iff for all  $\epsilon > 0$  there exists an  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $d(x_n, y_n) < \epsilon$ . The *Cauchy completion*  $\hat{X}$  of  $X$  is defined to be  $C/\sim$ . By mimicking the construction of the real numbers in class, show that:

- (a)  $\sim$  is an equivalence relation.
- (b) The function  $D : \hat{X} \times \hat{X} \rightarrow \mathbb{R}$  defined by

$$D([(x_n)], [(y_n)]) \mapsto \lim_{n \rightarrow \infty} d(x_n, y_n)$$

is well-defined, and  $D$  defines a metric on  $\hat{X}$ .

- (c) The metric space  $(\hat{X}, D)$  is Cauchy complete.
- (d) The function  $\iota : X \rightarrow \hat{X}$  defined by  $x \mapsto [(x)]$ , where  $(x)$  is the constant sequence all of whose terms are  $x$ , is injective, and  $D(\iota(x_1), \iota(x_2)) = d(x_1, x_2)$  for all  $x_1, x_2 \in X$ .
- (e) Let  $(Y, \rho)$  be a Cauchy complete metric space, and suppose  $f : X \rightarrow Y$  is a function such that  $\rho(f(x_1), f(x_2)) \leq d(x_1, x_2)$  for all  $x_1, x_2 \in X$ . Show that the function  $\hat{f} : \hat{X} \rightarrow Y$  defined by

$$\hat{f}([(x_n)]) = \lim_{n \rightarrow \infty} f(x_n)$$

is well-defined, and that  $\hat{f} \circ \iota = f$ .

**Note:** The idea is to mimic the proofs from class wherever possible. Issues only arise in those cases where we used the operations  $+$  and  $-$  on  $\mathbb{Q}$ , since we don't have those around on  $X$  now.

**Solution to (a):** Reflexivity and symmetry are clear. For transitivity, let  $(x_n), (y_n)$ , and  $(z_n)$  be Cauchy, and suppose  $(x_n) \sim (y_n)$  and  $(y_n) \sim (z_n)$ . Let  $\epsilon > 0$  be arbitrary. By assumption, there exists an  $N \in \mathbb{N} \setminus \{0\}$  such that for all  $n \geq N$ ,  $d(x_n, y_n) < \frac{\epsilon}{2}$  and  $d(y_n, z_n) < \frac{\epsilon}{2}$ . Hence for all  $n \geq N$ , we have

$$d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

**Solution to (b):** The first part is identical to the proof given in class.

Let  $(x_n)$  and  $(y_n)$  be Cauchy sequences, and let  $\epsilon > 0$  be arbitrary. There exists an  $N \in \mathbb{N} \setminus \{0\}$  such that for all  $n, m \geq N$ , we have  $d(x_n, x_m) < \frac{\epsilon}{2}$  and  $d(y_n, y_m) < \frac{\epsilon}{2}$ . Hence for all  $n, m \geq N$  we have:

$$d(x_n, y_n) \leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n) < \epsilon + d(x_m, y_m),$$

so

$$d(x_n, y_n) - d(x_m, y_m) < \epsilon$$

the same argument with  $n$  and  $m$  swapped shows that

$$d(x_m, y_m) - d(x_n, y_n) < \epsilon,$$

hence

$$|d(x_n, y_n) - d(x_m, y_m)| < \epsilon,$$

so  $(d(x_n, y_n))$  is Cauchy, and hence the limit

$$\lim_{n \rightarrow \infty} d(x_n, y_n)$$

actually exists since  $\mathbb{R}$  is Cauchy complete.

We need to check that  $D([(x_n)], [(y_n)])$  does not depend on the representatives  $(x_n)$  and  $(y_n)$ . Suppose that  $(x_n) \sim (x'_n)$  and  $(y_n) \sim (y'_n)$ . Let  $\epsilon > 0$  be arbitrary. Then there exists an  $N \in \mathbb{N} \setminus \{0\}$  such that for all  $n \geq N$ , we have  $d(x_n, x'_n) < \frac{\epsilon}{2}$  and  $d(y_n, y'_n) < \frac{\epsilon}{2}$ . Then for all  $n \geq N$ , we have

$$d(x_n, y_n) \leq d(x_n, x'_n) + d(x'_n, y'_n) + d(y'_n, y_n) < d(x'_n, y'_n) + \epsilon,$$

so

$$d(x_n, y_n) - d(x'_n, y'_n) < \epsilon,$$

and likewise

$$d(x'_n, y'_n) - d(x_n, y_n) < \epsilon,$$

so

$$|d(x_n, y_n) - d(x'_n, y'_n)| < \epsilon,$$

and hence the Cauchy sequences  $(d(x_n, y_n))$  and  $(d(x'_n, y'_n))$  are equivalent. Since both converge, they both have the same limit.

**Solution to (d):** Part (d) is straightforward, and is exactly like the case where  $X = \mathbb{Q}$ . I mention it here only because we will use it for (c).

**Outline of solution to (c):** This really can be done as for  $\mathbb{Q}$  with no changes. We first prove a lemma:

**Lemma 0.1.** *If  $(x_n)$  is a Cauchy sequence in  $X$ , then the sequence  $(\iota(x_n))$  converges in  $\hat{X}$  to  $[(x_n)]$ .*

The proof is identical to the one given in class for  $\mathbb{Q}$ . Now, let  $(a_n)$  be a Cauchy sequence in  $\hat{X}$ . We approximate the sequence  $(a_n)$  by elements of the form  $\iota(x_n)$ , and then show that  $(a_n)$  converges to  $[(x_n)]$ . To do this, for each  $n \in \mathbb{N} \setminus \{0\}$ , since  $a_n$  is a limit of elements “in  $X$ ” (by the lemma), we choose an  $x_n \in X$  such that  $D(a_n, \iota(x_n)) < \frac{1}{2^n}$ . The proof now that  $(\iota(x_n))$  is Cauchy and that  $(a_n)$  converges to  $[(x_n)]$  is identical to the proof in class for  $\mathbb{Q}$ .

**Outline to solution to (e):** This is new. Let  $(Y, \rho)$  be a Cauchy complete metric space, and suppose  $f : X \rightarrow Y$  is a function such that  $\rho(f(x_1), f(x_2)) \leq d(x_1, x_2)$  for all  $x_1, x_2 \in X$ . Then by Problem 2 on this HW, if  $(x_n)$  is a Cauchy sequence in  $X$ , then  $(f(x_n))$  is a Cauchy sequence in  $Y$ . We define the function  $\hat{f} : \hat{X} \rightarrow Y$  by

$$\hat{f}([(x_n)]) = \lim_{n \rightarrow \infty} f(x_n),$$

which exists since  $Y$  is Cauchy complete. I leave it to you to check that  $\hat{f}$  is well-defined and has the desired properties.