Math 74 Homework 13
Due Monday, November 24th

November 17, 2008

1. Let (x_n) and (y_n) be Cauchy sequences in $\mathbb{Q}$. Show that $(x_n) \sim (y_n)$ iff $(x_n^2 + y_n^2) \sim (2x_n y_n)$.

2. Let (X, d) and (Y, ρ) be metric spaces. Suppose that $f : X \to Y$ is a function such that $\rho(f(x_1), f(x_2)) \leq d(x_1, x_2)$ for all $x_1, x_2 \in X$. Show that if (x_n) is a Cauchy sequence in X, then $(f(x_n))$ is a Cauchy sequence in Y.

3. Let (X, d) be a metric space, and let (a_n) and (b_n) be convergent sequences in X such that $\lim_{n\to\infty} a_n \neq \lim_{n\to\infty} b_n$. Show that there exists a $\delta \in \mathbb{R}$, $\delta > 0$, and an $N \in \mathbb{N} \setminus \{0\}$ such that for all $n \geq N$, $d(a_n, b_n) > \delta$.

4. Let (X, d) be a metric space, and let C be the set of all Cauchy sequences in (X, d). Define an equivalence relation $\sim$ on C by $(x_n) \sim (y_n)$ iff for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $d(x_n, y_n) < \epsilon$. The *Cauchy completion* $\hat{X}$ of X is defined to be $C/\sim$. By mimicking the construction of the real numbers in class, show that:

   (a) $\sim$ is an equivalence relation.

   (b) The function $D : \hat{X} \times \hat{X} \to \mathbb{R}$ defined by
   $$D((x_n), (y_n)) \mapsto \lim_{n\to\infty} d(x_n, y_n)$$
   is well-defined, and D defines a metric on $\hat{X}$.

   (c) The metric space $(\hat{X}, D)$ is Cauchy complete.

   (d) The function $\iota : X \to \hat{X}$ defined by $x \mapsto [(x)]$, where (x) is the constant sequence all of whose terms are x, is injective, and $D(\iota(x_1), \iota(x_2)) = d(x_1, x_2)$ for all $x_1, x_2 \in X$.

(e) Let (Y, ρ) be a Cauchy complete metric space, and suppose $f : X \to Y$ is a function such that $\rho(f(x_1), f(x_2)) \leq d(x_1, x_2)$ for all $x_1, x_2 \in X$. Show that the function $\hat{f} : \hat{X} \to Y$ defined by

$$\hat{f}([(x_n)]) = \lim_{n \to \infty} f(x_n)$$

is well-defined, and that $\hat{f} \circ \iota = f$.