# Math 74 (Mini)-Homework 12: Solutions

November 17, 2008

Throughout, $C$ is the set of Cauchy sequences in $\mathbb{Q}$ (with respect to the Euclidean metric), $\sim$ is the equivalence relation on $C$ given by $(x_n) \sim (y_n)$ iff $\lim_{n\to\infty} x_n - y_n = 0$, and $R = C/{\sim}$.

1. Let $(x_n), (x'_n), (y_n)$, and $(y'_n)$ be Cauchy sequences in $\mathbb{Q}$, and suppose $(x_n) \sim (x'_n)$ and $(y_n) \sim (y'_n)$. Show that $(x_n y_n) \sim (x'_n y'_n)$.

   **Solution:** Let $\epsilon > 0$ be arbitrary. Since the sequence $(y_n)$ is Cauchy, it is bounded, so there exists an $M_1 \in \mathbb{N} \setminus \{0\}$ so that $|y_n - 0| = |y_n| \leq M_1$ for all $n \in \mathbb{N} \setminus 0$. Likewise there exists an $M_2 \in \mathbb{N} \setminus \{0\}$ such that $|x'_n| \leq M_2$ for all $n \in \mathbb{N} \setminus \{0\}$. Now, since $(x_n) \sim (x'_n)$ and $(y_n) \sim (y'_n)$, there exist $N_1, N_2 \in \mathbb{N} \setminus \{0\}$ so that $|x_n - x'_n| \leq \frac{\epsilon}{2M_2}$ for all $n \geq N_1$ and $|y_n - y'_n| \leq \frac{\epsilon}{2M_1}$ for all $n \geq N_2$. Hence for all $n \geq \max(N_1, N_2)$ we have that

   $$\begin{aligned} d(x_n y_n, x'_n y'_n) &= \left|x_n y_n - x'_n y'_n\right| \\ &= \left|x_n y_n - x'_n y_n + x'_n y_n - x'_n y'_n\right| \\ &\leq |y_n| \left|x_n - x'_n\right| + \left|x'_n\right| \left|y_n - y'_n\right| \\ &< M_2 \cdot \frac{\epsilon}{2M_2} + M_1 \cdot \frac{\epsilon}{2M_1} \\ &= \epsilon. \end{aligned}$$

   Thus $(x_n y_n) \sim (x'_n y'_n)$, as desired.

2. In class, we defined a relation $\leq$ on $R$ by $[(x_n)] \leq [(y_n)]$ iff $(x_n) \sim (y_n)$ or there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \leq y_n$. We checked that this was well-defined. Show that it's a partial order relation.

   **Solution:** Reflexivity: Let $[(x_n)]$ be any element of $R$. Then for all $n \geq 1$, $x_n \leq x_n$, hence $[(x_n)] \leq [(x_n)]$.

Transitivity: Let $[(x_n)], [(y_n)]$, and $[(z_n)]$ be any three elements of $R$, and suppose that $[(x_n)] \leq [(y_n)]$ and $[(y_n)] \leq [(z_n)]$. Then there exist $N_1, N_2 \in \mathbb{N} \setminus \{0\}$ such that for all $n \geq N_1$, we have $x_n \leq y_n$, and for all $n \geq N_2$, we have $y_n \leq z_n$. Let $N = \max(N_1, N_2)$. Then for all $n \geq N$, we have $x_n \leq y_n$ and $y_n \leq z_n$. By the transitivity of $\leq$ on $\mathbb{Q}$, we get $x_n \leq z_n$ for all $n \geq N$. Hence $[(x_n)] \leq [(z_n)]$.

Anti-symmetry: Let $[(x_n)]$ and $[(y_n)]$ in $R$ be arbitrary, and suppose that $[(x_n)] \leq [(y_n)]$ and $[(y_n)] \leq [(x_n)]$. Then there exist $N_1, N_2 \in \mathbb{N} \setminus \{0\}$ such that for all $n \geq N_1$, we have $x_n \leq y_n$, and for all $n \geq N_2$, we have $y_n \leq x_n$. Let $N = \max(N_1, N_2)$. Then for all $n \geq N$ we have $x_n \leq y_n$ and $y_n \leq x_n$, so $x_n = y_n$ by antisymmetry of $\leq$ on $\mathbb{Q}$. Hence for all $n \geq \mathbb{N}$, we have $d(x_n, y_n) = 0$, so in particular for any $\epsilon > 0$ we have for all $n \geq N$ that $d(x_n, y_n) < \epsilon$. Hence $(x_n) \sim (y_n)$, so $[(x_n)] = [(y_n)]$.

3. Let $i : \mathbb{Q} \to R$ be the function which sends a rational number $q$ to the class of the constant sequence $(q)$ (i.e. the sequence $q, q, q, q, \ldots$). Show that:

   (a) The function $i$ is injective.

   (b) For any $q, r \in \mathbb{Q}$, we have $q \leq r$ iff $i(q) \leq i(r)$.

   (c) Let $d$ be the metric on $R$ which we defined on Friday (we haven't checked that this is a metric yet). For any $q, r \in \mathbb{Q}$, show that $i(|q - r|) = d(i(q), i(r))$.

   **Solution to (a):** Suppose $i(q) = i(r)$ for some $q, r$ in $\mathbb{Q}$. Then by definition, $[(q)] = [(r)]$. hence $(q) \sim (r)$, so $q - r = \lim_{n\to\infty} q - r = 0$. Hence $q = r$.

   **Solution to (b):** By definition, $i(q) \leq i(r)$ iff $[(q)] \leq [(r)]$ iff there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $q \leq r$, which is to say, iff $q \leq r$.

   **Solution to (c):** We have $d(i(q), i(r)) = d([(q)], [(r)]) = (d(q, r)) = (|q - r|) = i(|q - r|)$.