Math 74 Homework 11: Selected Solutions

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1. Let $X = \{v, w, x, y\}$ be a set with four elements, and let $d : X \times X \to \mathbb{R}$ be a function such that

$$\begin{aligned} d(v,w) &= 2 \\ d(x,w) &= 1 \\ d(v,y) &= 5 \\ d(x,y) &= 1. \end{aligned}$$

Either prove that d cannot possibly be a metric, or give an example of such a d which is a metric.

Solution: The function d cannot possibly be a metric. Suppose d were a metric. Then we would have:

$$d(v,x) \leq d(v,w) + d(w,x) = 3$$

and we would also have

$$5 = d(v,y) \leq d(v,x) + d(x,y) = d(v,x) + 1,$$

hence $d(v,x) \geq 4$, which is a contradiction with $d(v,x) \leq 3$.

2. Let (X,d) be a metric space, and let (x_n) be a bounded sequence in (X,d). Show that for *every* $x \in X$ there is an $r \in \mathbb{R}$ such that $d(x_n, x) < r$ for all $n \in \mathbb{N}$.

Solution: Since (x_n) is bounded, there exists an $r \in \mathbb{R}$ and an $x \in X$ such that $d(x, x_n) < r$ for all $n \in \mathbb{N} \setminus \{0\}$. Let $y \in X$ be arbitrary, and let $R = r + d(x,y)$. Then for all $n \in \mathbb{N} \setminus \{0\}$, we have

$$d(y, x_n) \leq d(y,x) + d(x,x_n) < d(y,x) + r = R,$$

which is what we wanted.

3. Let (X, d) be a metric space, let $x \in X$ be a point, and let $r \in \mathbb{R}$ with $r > 0$ be arbitrary. We define the *open ball around x with radius r*, denoted $B(x, r)$, by:

$$B(x, r) := \{y \in X \mid d(x, y) < r\}.$$

   (a) Show that if $y \in B(x, r)$ then there exists a $t \in \mathbb{R}$, $t > 0$, such that $B(y, t) \subseteq B(x, r)$.

   (b) Show that if $y \in B(x, r) \cap B(x', r')$ then there exists a $t \in \mathbb{R}$, $t > 0$, such that $B(y, t) \subseteq B(x, r) \cap B(x', r')$.

Solution to a): Suppose $y \in B(x, r)$. Then $d(x, y) < r$ by definition. Let $t = \frac{r - d(x,y)}{2}$. Then

$$\begin{aligned} d(x, y) + t &= \frac{2d(x, y)}{2} + \frac{r - d(x, y)}{2} \\ &= \frac{r}{2} + \frac{d(x, y)}{2} \\ &< \frac{r}{2} + \frac{r}{2} = r. \end{aligned}$$

I claim that $B(y, t) \subseteq B(x, r)$. Let $z \in B(y, t)$ be arbitrary. Then $d(y, z) < t$ by definition. Hence we have

$$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + t < r.$$

So $z \in B(x, r)$, as desired.

Solution to b): Suppose $y \in B(x, r) \cap B(x', r')$. By (a), there exist $s, s' \in \mathbb{R}$, $s > 0$, $s' > 0$, such that

$$B(y, s) \subseteq B(x, r) \quad \text{and} \quad B(y, s') \subseteq B(x', r').$$

Let $t = \min(s, s')$. $B(y, t) \subseteq B(y, s) \subseteq B(x, r)$ and $B(y, t) \subseteq B(y, s') \subseteq B(x, r')$, hence $B(y, t) \subseteq B(x, r) \cap B(x', r')$, as desired.

4. Show that a sequence $((x_n, y_n))_{n \in \mathbb{N} \setminus \{0\}}$ in $\mathbb{R} \times \mathbb{R}$ converges to (x, y) with respect to the Euclidean metric if and only if (x_n) converges to x and (y_n) converges to y in $\mathbb{R}$ with respect to the Euclidean metric.

Solution: Suppose $((x_n, y_n))$ converges to (x, y) in $\mathbb{R} \times \mathbb{R}$ with respect to the Euclidean metric. Let $\epsilon > 0$ be arbitrary. Then there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$\sqrt{(x - x_n)^2 + (y - y_n)^2} < \epsilon.$$

Since both sides of this inequality are positive numbers, we have:

$$(x - x_n)^2 + (y - y_n)^2 < \epsilon^2.$$

Now, $(y - y_n)^2 \geq 0$, so we have

$$(x - x_n)^2 \leq (x - x_n)^2 + (y - y_n)^2 < \epsilon^2.$$

Taking square roots gives $|x - x_n| < \epsilon$, hence (x_n) converges to x in the Euclidean metric. The same proof shows that (y_n) converges to y.

On the other hand, suppose that (x_n) converges to x and (y_n) converges to y in the Euclidean metric. Then there exist $N_1, N_2 \in \mathbb{N}$ such that for all $n \geq N_1$ we have $|x - x_n| \leq \frac{\epsilon}{\sqrt{2}}$ and for all $n \geq N_2$ we have $|y - y_n| \leq \frac{\epsilon}{\sqrt{2}}$. Let $N = \max N_1, N_2$. Then for all $n \geq N$ we have

$$\sqrt{(x - x_n)^2 + (y - y_n)^2} < \sqrt{\left(\frac{\epsilon}{\sqrt{2}}\right)^2 + \left(\frac{\epsilon}{\sqrt{2}}\right)^2} = \epsilon,$$

so (x_n, y_n) converges to (x, y) in the Euclidean metric.