

Math 74 Homework 11
Due Monday, November 10th

November 2, 2008

1. Let X and Y be sets, let $f : X \rightarrow Y$ be a function, and let R be an equivalence relation on Y . Define a relation f^*R on X by $x_1(f^*R)x_2$ iff $f(x_1)Rf(x_2)$.
 - (a) Show that f^*R is an equivalence relation (we call f^*R the *pullback of the equivalence relation R along f*).
 - (b) Let P be a partition on Y , and define $f^*P := \{f^{-1}(A) \mid A \in P\}$, where $f^{-1}(A)$ denotes the *preimage* of A under f . Show that f^*P is a partition on X .
 - (c) Show that $X/(f^*R) = f^*(Y/R)$ (We say that “pullback commutes with forming quotients”).
2. Let $X = \{v, w, x, y\}$ be a set with four elements, and let $d : X \times X \rightarrow \mathbb{R}$ be a function such that

$$\begin{aligned}d(v, w) &= 2 \\d(x, w) &= 1 \\d(v, y) &= 5 \\d(x, y) &= 1.\end{aligned}$$

Either prove that d cannot possibly be a metric, or give an example of such a d which is a metric.

3. Let (X, d) be a metric space, and let (x_n) be a bounded sequence in (X, d) . Show that for *every* $x \in X$ there is an $r \in \mathbb{R}$ such that $d(x_n, x) < r$ for all $n \in \mathbb{N}$.
4. Let (X, d) be a metric space, let $x \in X$ be a point, and let $r \in \mathbb{R}$ with $r > 0$ be arbitrary. We define the *open ball around x with radius r* ,

denoted $B(x, r)$, by:

$$B(x, r) := \{y \in X \mid d(x, y) < r\}.$$

- (a) Show that if $y \in B(x, r)$ then there exists a $t \in \mathbb{R}$, $t > 0$, such that $B(y, t) \subseteq B(x, r)$.
 - (b) Show that if $y \in B(x, r) \cap B(x', r')$ then there exists a $t \in \mathbb{R}$, $t > 0$, such that $B(y, t) \subseteq B(x, r) \cap B(x', r')$.
5. Show that a sequence $((x_n, y_n))_{n \in \mathbb{N} \setminus \{0\}}$ in $\mathbb{R} \times \mathbb{R}$ converges to (x, y) with respect to the Euclidean metric if and only if (x_n) converges to x and (y_n) converges to y in \mathbb{R} with respect to the Euclidean metric.
6. Use the previous problem and the definition of convergence to show that the sequence (x_n) in \mathbb{R}^2 given by

$$x_n = \left(1 - \frac{1}{2^{n-1}}, \frac{n^2 + 1}{2n^2}\right)$$

converges to $(1, \frac{1}{2})$ in the Euclidean metric on \mathbb{R}^2 .