

Math 74 Homework 10

Due Monday, November 3rd

November 1, 2008

1. Let X be a set, and suppose that X is not finite. Using an inductive definition, show that there is an injective function $f : \mathbb{N} \rightarrow X$.
2. (The construction of \mathbb{Q} from \mathbb{Z})
 - (a) Let \sim be the relation on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ given by $(a, b) \sim (c, d)$ iff $ad = bc$. Show that \sim is an equivalence relation.
 - (b) Let $Q := \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) / \sim$. We'll write $[a, b]$ for $[(a, b)]$ in Q to make notation less confusing. Define operations $+$, $-$, \cdot on Q by

$$\begin{aligned}[a, b] + [c, d] &:= [ad + bc, bd] \\ [a, b] - [c, d] &:= [ad - bc, bd] \\ [a, b] \cdot [c, d] &:= [ac, bd],\end{aligned}$$

and define division on Q by $[a, b] / [c, d] := [ad, bc]$ for $c \neq 0$. Show that these operations are well-defined.

- (c) Show that there is a well-defined function $f : Q \rightarrow \mathbb{Q}$ defined by $f([a, b]) = \frac{a}{b}$, and that this function is a bijection. Show moreover that f preserves the operations defined in (b).
3. (Equivalence Relations and Partitions are in Bijective Correspondence)
 - (a) Let X be a set, and let $P \subseteq P(X)$ be a partition of X . Define a relation R_P on X by aR_Pb iff there exists an $A \in P$ such that $a \in A$ and $b \in A$. Show that R_P is an equivalence relation.
 - (b) Now let R be an arbitrary equivalence relation on X . We showed in class that the set $X/R \subseteq P(X)$ is a partition of X . Show that the functions $R \mapsto X/R$ and $P \mapsto R_P$ are inverse bijections between the set of equivalence relations on X and the set of partitions of X .

4. Let X be a set, let \leq be a partial order relation on X , and let $Y \subseteq X$. Show that if a least upper bound for Y exists, then it is unique. Do the same for greatest lower bounds for Y .
5. Let $a, b \in \mathbb{N} \setminus \{0\}$. Let $p_1 < p_2 < \dots < p_n \in \mathbb{N}$ be all of the prime numbers appearing in the prime factorizations of either a or b . Then there exist unique numbers $r_1, r_2, \dots, r_n, s_1, s_2, \dots, s_n \in \mathbb{N}$ such that

$$a = p_1^{r_1} \cdot p_2^{r_2} \cdot \dots \cdot p_n^{r_n} \quad \text{and} \quad b = p_1^{s_1} \cdot p_2^{s_2} \cdot \dots \cdot p_n^{s_n}.$$

- (a) Show that a divides b if and only if $r_i \leq s_i$ for all $i \in \{1, \dots, n\}$.
- (b) For each $i \in \{1, \dots, n\}$, let $\min(r_i, s_i)$ denote the smaller of r_i and s_i (so $\min(1, 2) = 1$, for example). Show that

$$\gcd(a, b) = p_1^{\min(r_1, s_1)} \cdot p_2^{\min(r_2, s_2)} \cdot \dots \cdot p_n^{\min(r_n, s_n)}.$$

- (c) Calculate the prime factorizations of 38700 and 32760. Use part (b) to calculate $\gcd(38700, 32760)$.
6. Find an inverse to 8 mod 45. Use this to find a solution to the equation $8x = 13 \pmod{45}$.
7. Show that the equation $x^6 + 9x^2 + 1 = 0$ has no integer solutions.