Math 74 Final Exam: Solutions

December 16, 2008

1. (a) Let (X, d) be a metric space, and let (x_n) be a sequence in (X, d). Define what it means for (x_n) to be *convergent*.

   (b) Use quantifier negation to give a definition of "(x_n) is not convergent."

   (c) Pick your favorite metric space (X, d) and your favorite non-convergent sequence (x_n) in (X, d). Use your definition from part (b) to prove that your sequence (x_n) is not convergent.

   Solution to (a): We say (x_n) is *convergent* if there exists an $x \in X$ such that for all $\epsilon \in \mathbb{R}$ with $\epsilon > 0$ there exists an $N \in \mathbb{N} \setminus \{0\}$ such that for all $n \geq N$ we have $d(x_n, x) < \epsilon$.

   Solution to (b): For (x_n) to not be convergent means that for all $x \in X$ there exists an $\epsilon \in \mathbb{R}$ with $\epsilon > 0$ such that for all $N \in \mathbb{N} \setminus \{0\}$ there exists an $n \geq N$ such that $d(x_n, x) \geq \epsilon$.

   Solution to (c): Since so many people picked it: take the metric space $(\mathbb{R}, d)$ of the real numbers with the Euclidean metric. Let (x_n) be the sequence in $\mathbb{R}$ given by $x_n = (-1)^n$.

   Then I claim that (x_n) is not convergent. Let $x \in \mathbb{R}$ be arbitrary, let $\epsilon = 1$, and let $N \in \mathbb{N}$ be arbitrary. If $x \geq 0$, then there is an $n \geq N$ such that n is odd, and hence $x_n = -1$. Then $d(x, x_n) = |x + 1| = x + 1 \geq 1 = \epsilon$. On the other hand, if $x < 0$, then there is an $n \geq N$ such that n is even, and hence $x_n = 1$. Then $d(x, x_n) = |x - 1| = 1 - x > 1 = \epsilon$. Hence (x_n) is not convergent.

2. Let X be a set. We call a function $d : X \times X \to \mathbb{R}$ a *pseudometric* if the following hold:

   (a) For all $x \in X$, we have $d(x, x) = 0$.
   (b) For all $x, y, \in X$, we have $d(x, y) = d(y, x)$.
   (c) For all $x, y, z \in X$ we have $d(x, y) \le d(x, z) + d(z, y)$.

   Let d be a pseudometric on X. Do the following:

   (a) Explain how this definition differs from the definition of a *metric*.
   (b) Show that $d(x, y) \ge 0$ for all $x, y \in X$.
   (c) Let $x, y \in X$. Show that if $d(x, y) = 0$ then $d(x, z) = d(y, z)$ for all $z \in X$.
   (d) Show that the relation $\sim$ on X given by $x \sim y$ iff $d(x, y) = 0$ is an equivalence relation.

Solution to (a): If d were a metric, we would require that $d(x, y) = 0$ *if and only if* $x = y$. For a pseudometric, we are asking that if $x = y$, then $d(x, y) = 0$, but *not* necessarily the other way around.

Solution to (b): Let $x, y \in X$ be arbitrary. Then

$$0 = d(x, x) \le d(x, y) + d(y, x) = d(x, y) + d(x, y) = 2d(x, y).$$

Dividing both sides by 2, we have $d(x, y) \ge 0$.

Solution to (c): Let $x, y, z \in X$ be arbitrary, and suppose that $d(x, y) = 0$. Then we have

$$d(x, z) \le d(x, y) + d(z, y) = 0 + d(y, z)$$

and

$$d(y, z) \le d(y, x) + d(x, z) = 0 + d(x, z),$$

hence $d(x, z) \le d(y, z)$ and $d(y, z) \le d(x, z)$, thus $d(x, z) = d(y, z)$.

Solution to (d): **Reflexivity:** By axiom (a) for a pseudometric, we have that $d(x, x) = 0$ for all $x \in X$, hence $x \sim x$ for all $x \in X$.

Symmetry: Suppose $x \sim y$. Then $d(x, y) = 0$. By axiom (b) for a pseudometric, we have that $0 = d(x, y) = d(y, x)$, hence $y \sim x$.

Transitivity: Suppose $x \sim y$ and $y \sim z$. Then $d(x, y) = 0 = d(y, z)$. By part (c) above, we have that since $d(x, y) = 0$ it follows that $d(x, z) = d(y, z) = 0$. Thus $x \sim z$.

3. Show that there does not exist a rational number q such that $q^3 = 2$.

Solution: Suppose that $q^3 = 2$ for some rational number q. Certainly $q > 0$. Write $q = \frac{a}{b}$ for some $a, b \in \mathbb{N} \setminus \{0\}$. Then $\left(\frac{a}{b}\right)^3 = 2$, hence $a^3 = 2b^3$. Let

$$a = 2^r \cdot p_1^{r_1} \cdot p_2^{r_2} \cdots p_n^{r_n}$$

be the prime factorization of a, where $2 < p_1 < \cdots < p_n$ (and possibly $r = 0$), and let

$$b = 2^s \cdot q_1^{s_1} \cdot q_2^{s_2} \cdots q_m^{s_m}$$

be the prime factorization of b, where $2 < q_1 < \cdots < q_m$ (and possibly $s = 0$).

Then from $a^3 = 2b^3$, we have

$$2^{3r} \cdot p_1^{3r_1} \cdot p_2^{3r_2} \cdots p_n^{3r_n} = 2^{3s+1} \cdot q_2^{3s_2} \cdots q_m^{3s_m}.$$

By the uniqueness of prime factorization, we have that $3r = 3s + 1$. Hence $3(r - s) = 1$, so 3 divides 1, which is absurd. Hence our assumption was false and no such q exists.

4. Show that

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

for all $n \in \mathbb{N}$.

Solution: By induction. For $n = 0$, both sides are equal to 0. Suppose

$$\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$$

for some $k \in \mathbb{N}$. We wish to show this formula holds with k replaced by $k+1$. We have:

$$\begin{aligned} \sum_{i=1}^{k+1} \frac{1}{i(i+1)} &= \frac{1}{(k+1)(k+2)} + \sum_{i=1}^{k} \frac{1}{i(i+1)} \\ &= \frac{1}{(k+1)(k+2)} + \frac{k}{k+1} \\ &= \frac{1+k(k+2)}{(k+1)(k+2)} \\ &= \frac{(k+1)(k+1)}{(k+1)(k+2)} = \frac{k+1}{(k+1)+1}, \end{aligned}$$

as desired.

5. Let ρ be the discrete metric on $\mathbb{R}$, let d be the Euclidean metric on $\mathbb{R}$, and let

$$f : (\mathbb{R}, \rho) \to (\mathbb{R}, d)$$

be the function defined by $f(r) = r$ for all $r \in \mathbb{R}$.

(a) Is f continuous? Prove your answer.

(b) Is f^{-1} continuous? Prove your answer.

Solution 1 to (a): The function f is continuous. Let $U \subseteq \mathbb{R}$ be any open set. Then $f^{-1}(U)$ is open, since *all* subsets of a discrete space are open. Thus f is continuous.

Solution 2 to (a): Let (x_n) in $(\mathbb{R}, \rho)$ be a convergent sequence, and let $x = \lim_n x_n$. Then (x_n) is eventually constant since X is discrete, i.e. there exists an $N \in \mathbb{N}$ such that $x_n = x$ for all $n \geq N$. Hence also $f(x_n) = f(x) = x$ for all $n \geq N$. Thus $f(x_n)$ is eventually the constance sequence x, hence is convergent to $x = f(x)$. Thus f is continuous.

Solution 1 to (b): The function $g = f^{-1}$ is not continuous. The set $\{0\} \subseteq \mathbb{R}$ is an open set in $(\mathbb{R}, \rho)$ (it's the open ball of radius 1/2 around 0), but $g^{-1}(\{0\}) = \{0\}$ is not an open subset of $(\mathbb{R}, d)$, since for all $r > 0$ we have that $B(0, r) \nsubseteq \{0\}$.

Solution 2 to (b): Let (x_n) in $(\mathbb{R}, d)$ be the sequence $x_n = \frac{1}{n}$. Then (x_n) converges to 0. But the sequence $(f^{-1}(x_n)) = (\frac{1}{n})$ is *not* convergent in $(\mathbb{R}, \rho)$, since it is not eventually constant. Hence f^{-1} is not continuous.

6. Let $A \subseteq \mathbb{R}$ be a closed subset of $\mathbb{R}$ in the Euclidean metric. Show that $A \times \{0\} \subseteq \mathbb{R}^2$ is a closed subset of $\mathbb{R}^2$ in the Euclidean metric.

Solution 1: Let $p_1, p_2 : \mathbb{R}^2 \to \mathbb{R}$ be the projections $p_1(x, y) = x$, $p_2(x, y) = y$. We proved in class that these functions are continuous. Now, A and $\{0\}$ are both closed subsets of $\mathbb{R}$, hence $p_1^{-1}(A)$ and $p_2^{-1}(\{0\})$ are both closed subsets of $\mathbb{R}^2$. Thus

$$A \times \{0\} = p_1^{-1}(A) \cap p_2^{-1}(\{0\})$$

is also closed.

Solution 2: Let (x_n, y_n) be a sequence in $A \times \{0\}$ which converges to $(x, y) \in \mathbb{R}^2$. Then (x_n) converges to x and (y_n) converges to y. Then since $(x_n, y_n) \in A \times \{0\}$ we have $x_n \in A$ and $y_n = 0$ for all $n \in \mathbb{N} \setminus \{0\}$. Now, since A is closed and x is the limit of the sequence (x_n) in A, we have that $x \in A$. And since $y_n = 0$ for all n and y is the limit of (y_n), we have that $y = 0$. Hence $(x, y) = (x, 0) \in A \times \{0\}$, and so $A \times \{0\}$ contains all its limit points.

7. Do **ONE** of the following:

   (a) Show that there are infinitely many primes $p \in \mathbb{N}$ such that $p+2$ is also a prime.

   (b) Show that every even $n \in \mathbb{N}$ with $n > 2$ can be written as $n = p+q$ for some primes p and q.

   (c) Let $p \in \mathbb{N}$ be a prime and let $a \in \mathbb{Z}$ be arbitrary. Show that if $a^2 \equiv 1 (\mathrm{mod}\ p)$ then either $a \equiv 1 (\mathrm{mod}\ p)$ or $a \equiv -1 (\mathrm{mod}\ p)$.

Solution to (c): Suppose that $a^2 \equiv 1 (\mathrm{mod}\ p)$. By definition, $p \mid (a^2 - 1)$. Hence $p \mid (a+1)(a-1)$. Since p is prime, either $p \mid (a+1) = (a - (-1))$ or $p \mid (a-1)$. Thus $a \equiv -1 (\mathrm{mod}\ p)$ or $a \equiv 1 (\mathrm{mod}\ p)$.

8. Write a short ($\leq$ 2 page) essay entitled "Using equivalence relations to construct new mathematical objects." You do not need to include proofs, and your essay should look like an "English class essay," i.e. should be in usual paragraph format. Include examples from class to illustrate your points.

   Note: I will post a sample essay soon! Thanks so much for sticking with the class to the end; I hope you learned a lot and have at least gained some appreciation for the world of proof-based mathematics. I really enjoyed teaching all of you and feel very fortunate for having had the opportunity to do so. Have a great break and best of luck with your classes next semester!