Math 74 Final Exam Practice Problems: Selected Solutions

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Easier Problems

1. Show that $\mathbb{Z}$ is a closed subset of $\mathbb{R}$ in the Euclidean metric.

   Solution 1: We will show that $\mathbb{Z}$ is closed by showing it contains all its limit points. Let (x_n) be a sequence in $\mathbb{Z}$ which converges in $\mathbb{R}$. Then (x_n) is Cauchy, so there is an $N \in \mathbb{N}$ such that $d(x_n, x_m) < 1/2$ for all $n, m \geq N$. Now, when $a, b \in \mathbb{Z}$, we have $d(a, b) \in \mathbb{N}$. Since the only natural number less than $1/2$ is 0, we conclude that $d(x_n, x_m) = 0$ for all $n, m \geq N$, hence $x_n = x_m$ for all $n, m \geq N$, i.e. $x_n = x_N$ for all $n \geq N$. Thus $\lim_n x_n = x_N \in \mathbb{Z}$. Hence $\mathbb{Z}$ contains all its limit points.

   Solution 2: We instead show that $\mathbb{R} \setminus \mathbb{Z}$ is open. Let $r \in \mathbb{R} \setminus \mathbb{Z}$ be arbitrary. Then there exists an $n \in \mathbb{Z}$ such that $n < r < n + 1$. For all $m \in \mathbb{N}$, we either have $m \leq n$, in which case $d(m, r) \geq d(n, r) = r - n$, or else we have $m \geq n+1$, in which case $d(m, r) \geq d(n+1, r) = n+1-r$, so if we let $\delta = \min(r-n, n+1-r)$, then $d(m, r) \geq \delta$ for all $m \in \mathbb{N}$. Hence $B(r, \delta) \cap \mathbb{Z} = \emptyset$, i.e. $B(r, \delta) \subseteq \mathbb{R} \setminus \mathbb{Z}$. Thus $\mathbb{R} \setminus \mathbb{Z}$ is open in $\mathbb{R}$.

2. Let (X, d) be a metric space and let $Y \subseteq X$ be any subset. Since $Y \times Y \subseteq X \times X$, the function $d : X \times X \to \mathbb{R}$ restricts to a function $d_Y : Y \times Y \to \mathbb{R}$. Show that (Y, d_Y) is a metric space.

   Solution: We check the three axioms for a metric; all follow immediately from the same axioms for d:

   (a) Let $x, y \in Y$ be arbitrary. We have $d_Y(x, y) = 0$ if and only if $d(x, y) = 0$ if and only if $x = y$, since d is a metric on X.

(b) Let $x, y \in Y$ be arbitrary. We have $d_Y(x,y) = d(x,y) = d(y,x) = d_Y(y,x)$.

(c) Let $x, y, z \in Y$ be arbitrary. We have $d_Y(x,y) = d(x,y) \leq d(x,z) + d(z,y) = d_Y(x,z) + d_Y(z,y)$.

3. For each of the three definitions of continuity, give a proof that the identity function $1_X : X \to X$ on any metric space (X, d) is a continuous function.

Preservation of limits: Let (x_n) be an arbitrary convergent sequence in X, and let $x = \lim_n x_n$. Then we have

$$\lim_n 1_X(x_n) = \lim_n x_n = x = 1_X(x),$$

hence 1_X is continuous.

Epsilon-Delta: Let $x \in X$ and $\epsilon > 0$ be arbitrary. Let $\delta = \epsilon$. Then we have that if $d(x,y) < \delta$ then $d(1_X(x), 1_X(y)) = d(x,y) < \delta = \epsilon$. Hence 1_X is continuous.

Open Sets: Let $U \subseteq X$ be an arbitrary open set. Then $1_X^{-1}(U) = U$ is open. Hence 1_X is continuous.

4. Using the definition of convergence, show that the sequence (s_n) in $\mathbb{R}$ defined by

$$s_n = \sum_{i=1}^{n} \frac{1}{2^n}$$

converges to 1.

Solution: It is easy to show by induction that $s_n = \frac{2^n - 1}{2^n}$. Alternatively, you can use the fact that

$$\sum_{i=1}^{n} \frac{1}{2^n} = \frac{\frac{1}{2} - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}} = 1 - \frac{1}{2^n} = \frac{2^n - 1}{2^n}.$$

Now, we have that

$$d(s_n, 1) = |s_n - 1| = \left| \frac{2^n - 1}{2^n} - 1 \right| = \frac{1}{2^n}.$$

Now, let $\epsilon > 0$ be arbitrary. Let $N \in \mathbb{N}$ be such that $2^N > \frac{1}{\epsilon}$. Then for all $n \geq N$, we have $2^n \geq 2^N > \frac{1}{\epsilon}$, hence $\epsilon > \frac{1}{2^n}$. Hence for all $n \geq N$ we have

$$d(s_n, 1) = \frac{1}{2^n} < \epsilon.$$

Thus (s_n) converges to 1.

5. Let (X, d) and (Y, ρ) be metric spaces, and let $f : X \to Y$ be a function. Use quantifier negation and the epsilon-delta definition to give a careful definition of what it means for f to **not** be continuous. Use this to show that the function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

is not continuous.

Solution: Definition of "f is not continuous": f is not continuous if there exists an $x \in X$ and an $\epsilon > 0$ such that for all $\delta > 0$ there exists a $y \in X$ such that $d(x, y) < \delta$ and $\rho(f(x), f(y)) \geq \epsilon$.

To apply this to the given f, let $x = 0$ and let $\epsilon = 1/2$. Let $\delta > 0$ be arbitrary. Then $B(x, \delta)$ contains some positive numbers (since $x = 0$), i.e. there is a $y \in \mathbb{R}$ such that $d(x, y) < \delta$ and $y > 0$. Hence $d(f(x), f(y)) = d(0, 1) = 1 > \epsilon$. Thus f is not continuous.

6. Let $a, b \in \mathbb{Z}$, and suppose that $ab \equiv 2 (\text{mod } 4)$. Show that either a or b is even, but not both.

Solution: By definition, $ab \equiv 2$ if and only if there exists an $n \in \mathbb{Z}$ such that $ab - 2 = 4n$, i.e. $ab = 2 + 4n$. Hence ab is even, i.e. $2 \mid ab$. Thus either $2 \mid a$ or $2 \mid b$ since 2 is prime, so either a or b is even. Suppose both a and b were even. Then $a = 2k$ and $b = 2\ell$ for some $k, \ell \in \mathbb{Z}$. Hence $ab = 4k\ell$, so $ab \equiv 0 (\text{mod } 4)$, contradicting our assumption.

Medium Problems

7. Use the epsilon-delta definition of continuity to show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ is continuous. Give another (easier) proof of this fact using the techniques developed in class.

Solution: Let $x \in X$ and $\epsilon > 0$ be arbitrary. Let $\delta = \min(1, \frac{\epsilon}{3|x|^2+3|x|+1})$. Then if $d(x, y) < \delta$, we have in particular that $d(x, y) < 1$, hence

$$|y| = |y - x + x| \leq d(x, y) + |x| < 1 + |x| .$$

Then we have that

$$\left|x^2 + xy + y^2\right| \leq |x|^2 + |x|\,|y| + |y|^2 < 3\,|x|^2 + 3\,|x| + 1.$$

Hence we have

$$\begin{aligned} |f(x) - f(y)| &= \left|x^3 - y^3\right| \\ &= |x - y|\left|x^2 + xy + y^2\right| \\ &< \frac{\epsilon}{3\,|x|^2 + 3\,|x| + 1} \cdot (3\,|x|^2 + 3\,|x| + 1) = \epsilon. \end{aligned}$$

Thus f is continuous.

Note: Of course, the above proof leaves out how the solution was actually obtained. Make sure you understand how you would actually come up with this proof!

Easy solution: We know the identity function $1_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ defined by $1_{\mathbb{R}}(x) = x$ is continuous. We have $f = 1_{\mathbb{R}} \cdot 1_{\mathbb{R}} \cdot 1_{\mathbb{R}}$. Since products of continuous functions are continuous, f is continuous.

8. Let $A \subseteq \mathbb{R}$ be a closed subset (in the Euclidean metric) and let $r \in \mathbb{R}$ be arbitrary. Show that the set $A + r := \{a + r \mid a \in A\}$ is closed.

Solution: Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(a) = a - r$. Then f is continuous (it's the identity function minus a constant function). Now, note that $A + r = f^{-1}(A)$. Indeed, if $x \in A + r$, then $x = a + r$ for some $a \in A$, hence $f(x) = a + r - r = a \in A$, so $x \in f^{-1}(A)$. On the other hand if $x \in f^{-1}(A)$, then $f(x) = x - r$ is in A, hence $x = x - r + r \in A + r$. Since f is continuous and A is closed, $f^{-1}(A) = A + r$ is closed.

9. Let (X, d) and (Y, ρ) be metric spaces, let $f : X \to Y$ be a continuous function, and let $A \subseteq X$ be an arbitrary subset. Show that $f(\bar{A}) \subseteq \overline{f(A)}$ (where the bars here stand for closure). Give an example that shows that $f(\bar{A})$ can be a *proper* subset of $\overline{f(A)}$.

Solution 1: We want to show that $f(\bar{A}) \subseteq \overline{f(A)}$. Let $a \in \bar{A}$ be arbitrary. Then a is a limit point of A, so there exists a sequence (a_n) in A such that $a = \lim_n a_n$. Hence $f(a) = \lim_n f(a_n)$ since f is continuous. And $f(a_n) \in f(A)$ for all n. Hence $f(a)$ is a limit point of $f(A)$, i.e. $f(a) \in \overline{f(A)}$.

Solution 2: We have that $f(A) \subseteq \overline{f(A)}$, hence

$$A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(\overline{f(A)}).$$

Now, since f is continuous and $\overline{f(A)}$ is closed, $f^{-1}(\overline{f(A)})$ is closed, and the above shows this set contains A. Hence also $\bar{A} \subseteq f^{-1}(\overline{f(A)})$. Thus $f(\bar{A}) \subseteq \overline{f(A)}$.

Counterexample to Equality: Let $f : \mathbb{Q} \to \mathbb{R}$ be the natural inclusion function $f(q) = q$. Give $\mathbb{Q}$ and $\mathbb{R}$ each the Euclidean topology. Then $f(\overline{\mathbb{Q}}) = f(\mathbb{Q}) = \mathbb{Q}$, and $\overline{f(\mathbb{Q})} = \overline{\mathbb{Q}} = \mathbb{R} \neq \mathbb{Q}$.

10. Let X and Y be sets, let $f : X \to Y$ be a function, and let A be a subset of Y. Show that $f(f^{-1}(A)) = f(X) \cap A$.

Solution: $(f(f^{-1}(A)) \subseteq f(X) \cap A)$:
Let $y \in f(f^{-1}(A))$ be arbitrary. Then there exists an $x \in f^{-1}(A) \subseteq X$ such that $f(x) = y$. Hence in particular $y \in f(X)$. On the other hand, since $x \in f^{-1}(A)$, we have that $f(x) \in A$, i.e. $y \in A$. Hence $y \in f(X) \cap A$.

$(f(f^{-1}(A)) \supseteq f(X) \cap A)$:
Let $y \in f(X) \cap A$ be arbitrary. Then $y \in f(X)$, so there is an $x \in X$ such that $f(x) = y$. Now, since $y \in A$, we have $x \in f^{-1}(A)$, hence $y = f(x) \in f(f^{-1}(A))$.

11. Let $a \in \mathbb{N} \setminus \{0\}$ be arbitrary, and let

$$a = p_1^{r_1} \cdot p_2^{r_2} \cdots p_n^{r_n}$$

be the prime factorization of a. Find a formula in terms of the r_i for the number of distinct natural numbers which divide a. Prove that your answer is correct.

Solution: Let $b \in \mathbb{N} \setminus \{0\}$ be another natural number, and suppose that $b \mid a$. If p is a prime and $p \mid b$, then $p \mid a$, so $p = p_i$ for some $i \in \{1, \ldots, n\}$. Hence the only primes that divide b are $p_1, \ldots, p_n$. Hence the prime factorization of b is

$$b = p_1^{s_1} \cdots p_n^{s_n}$$

for some $s_1, \ldots, s_n \in \mathbb{N}$. Moreover, we have that $b \mid a$ if and only if $s_i \leq r_i$ for all $i \in \{1, \ldots, n\}$. Hence the natural numbers dividing a are precisely the numbers

$$p_1^{s_1} \cdots p_n^{s_n}$$

where $0 \leq s_i \leq r_i$ for all i. By unique factorization, each different n-tuple $(s_1, \ldots, s_n)$ gives a different natural number, so the number of distinct natural numbers which divide a is the same as the number of n-tuples $(s_1, \ldots, s_n)$ with $0 \leq s_i \leq r_i$. This is the same as the cardinality of the set

$$\{0, 1, \ldots, r_1\} \times \{0, 1, \ldots, r_2\} \times \cdots \times \{0, 1 \ldots, r_n\},$$

which is

$$(r_1 + 1) \cdot (r_2 + 1) \cdots (r_n + 1) = \prod_{i=1}^{n} (r_i + 1).$$

12. Prove the *binomial theorem*: if $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$, then

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}.$$

Solution: We proceed by induction. The case $n = 0$ is:

$$(a+b)^0 = 1 = \sum_{i=0}^{0} \binom{0}{0} a^0 b^0.$$

In case you don't like that, the case $n = 1$ says:

$$(a+b)^1 = a + b = \binom{1}{0} a^0 b^1 + \binom{1}{1} a^1 b^0.$$

Now, suppose the statement is true for $n = k$, i.e. suppose that

$$(a+b)^k = \sum_{i=0}^{k} \binom{k}{i} a^i b^{k-i}.$$

We want to show it is true for $n = k + 1$. We have

$$\begin{aligned}
(a+b)^{k+1} &= (a+b)(a+b)^k \\
&= (a+b) \sum_{i=0}^{k} \binom{k}{i} a^i b^{k-i} \\
&= \sum_{i=0}^{k} \binom{k}{i} a^{i+1} b^{k-i} + \sum_{i=0}^{k} \binom{k}{i} a^i b^{k-i+1} \\
&= a^{k+1} + b^{k+1} + \sum_{i=0}^{k-1} \binom{k}{i} a^{i+1} b^{k-i} \sum_{i=1}^{k} \binom{k}{i} a^i b^{k-i+1}
\end{aligned}$$

Here the last equality comes from "peeling off the $i = k$ term" from the first sum and "peeling off the $i = 0$ term" from the second sum. Next we change the numbering on the second sum, and continue to compute:

$$\begin{aligned}
&= a^{k+1} + b^{k+1} + \sum_{i=0}^{k-1} \binom{k}{i} a^{i+1} b^{k-i} + \sum_{i=0}^{k-1} \binom{k}{i+1} a^{i+1} b^{k-i} \\
&= a^{k+1} + b^{k+1} + \sum_{i=0}^{k-1} \left(\binom{k}{i} + \binom{k}{i+1} \right) a^{i+1} b^{k-i} \\
&= a^{k+1} + b^{k+1} + \sum_{i=0}^{k-1} \binom{k+1}{i+1} a^{i+1} b^{k-i} \\
&= a^{k+1} + b^{k+1} + \sum_{i=1}^{k} \binom{k+1}{i} a^{i} b^{k-i+1} \\
&= \sum_{i=0}^{k+1} \binom{k+1}{i} a^{i} b^{k-i+1},
\end{aligned}$$

which was what we wanted.