Math 74 Final Exam Practice Problems

December 9, 2008

Easier Problems

1. Show that $\mathbb{Z}$ is a closed subset of $\mathbb{R}$ in the Euclidean metric.

2. Let (X, d) be a metric space and let $Y \subseteq X$ be any subset. Since $Y \times Y \subseteq X \times X$, the function $d : X \times X \to \mathbb{R}$ restricts to a function $d_Y : Y \times Y \to \mathbb{R}$. Show that (Y, d_Y) is a metric space.

3. For each of the three definitions of continuity, give a proof that the identity function $1_X : X \to X$ on any metric space (X, d) is a continuous function.

4. Using the definition of convergence, show that the sequence (s_n) in $\mathbb{R}$ defined by
$$s_n = \sum_{i=1}^{n} \frac{1}{2^n}$$
converges to 1.

5. (Not a problem, a "friendly suggestion.") Review the construction of $\mathbb{Z}, \mathbb{Q}$, and $\mathbb{R}$.

6. Let (X, d) and (Y, ρ) be metric spaces, and let $f : X \to Y$ be a function. Use quantifier negation and the epsilon-delta definition to give a careful definition of what it means for f to **not** be continuous. Use this to show that the function $f : \mathbb{R} \to \mathbb{R}$,
$$f(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$
is not continuous.

7. Let $a, b \in \mathbb{Z}$, and suppose that $ab \equiv 2 (\text{mod } 4)$. Show that either a or b is even, but not both.

Medium Problems

8. Use the epsilon-delta definition of continuity to show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ is continuous. Give another (easier) proof of this fact using the techniques developed in class.

9. Let $A \subseteq \mathbb{R}$ be a closed subset (in the Euclidean metric) and let $r \in \mathbb{R}$ be arbitrary. Show that the set $A + r := \{a + r \mid a \in A\}$ is closed.

10. Let (X, d) and (Y, ρ) be metric spaces, let $f : X \to Y$ be a continuous function, and let $A \subseteq X$ be an arbitrary subset. Show that $f(\bar{A}) \subseteq \overline{f(A)}$ (where the bars here stand for closure). Give an example that shows that $f(\bar{A})$ can be a *proper* subset of $\overline{f(A)}$.

11. Prove the *squeeze theorem*: if $(x_n), (y_n)$, and (z_n) are sequences in $\mathbb{R}$ such that:

    (a) Both (x_n) and (z_n) converge, and $\lim_n x_n = \lim_n z_n$, and

    (b) For all $n \in \mathbb{N} \setminus \{0\}$, we have $x_n \leq y_n \leq z_n$,

    then the sequence (y_n) also converges, and $\lim_n y_n = \lim_n x_n$.

12. Let X and Y be sets, let $f : X \to Y$ be a function, and let A be a subset of Y. Show that $f(f^{-1}(A)) = f(X) \cap A$.

13. Let $a \in \mathbb{N} \setminus \{0\}$ be arbitrary, and let

$$a = p_1^{r_1} \cdot p_2^{r_2} \cdots p_n^{r_n}$$

    be the prime factorization of a. Find a formula in terms of the r_i for the number of distinct natural numbers which divide a. Prove that your answer is correct.

14. Prove the *binomial theorem*: if $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$, then

$$(a + b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}.$$

15. Show that if p is prime and $1 \leq i \leq p - 1$, then $\binom{p}{i}$ is divisible by p. Give an example that shows that this can be false if p is not prime.

Harder Problems

16. Let (X, d) be a metric space. Suppose that for each $n \in \mathbb{N} \setminus \{0\}$, $Y_n \subseteq X$ is a subset of X which is Cauchy complete (with the metric restricted from X). Must $\bigcup_{n \in \mathbb{N}} Y_n$ be Cauchy complete? What about $\bigcap_{n \in \mathbb{N}} Y_n$?

17. Show that $\mathbb{R}$ is a connected metric space. [Outline: start by assuming that $U \subseteq \mathbb{R}$ is both open and closed, and that $U \neq \mathbb{R}$ and $U \neq \emptyset$. Let $V = \mathbb{R} \setminus U$, and let $x \in U$ be arbitrary. Show that for some $r \in \mathbb{R}$, $r > 0$, either $(x, x+r) \cap V$ or $(x-r, x) \cap V$ is nonempty. Assuming $(x, x+r) \cap V$ is nonempty, let s be the greatest lower bound on the set $\{r \in \mathbb{R} \mid r > 0, (x, x+r) \cap V \neq \emptyset\}$. Show that $x + s$ is an element of both U and V.]

18. Let $a \in \mathbb{R}$ be the least upper bound for the set $\{q \in \mathbb{Q} \mid q^2 \leq 2\} \subseteq \mathbb{R}$. Explain why a exists, and show that $a^2 = 2$.