Math 74 Final Exam

December 16th, 2008

Name ______________________ SID _________________

Question	Score	Possible
1		10
2		11
3		6
4		6
5		10
6		6
7		5
8		7
$\sum$		61

1. (a) Let (X, d) be a metric space, and let (x_n) be a sequence in (X, d). Define what it means for (x_n) to be *convergent*.

   (b) Use quantifier negation to give a definition of "(x_n) is not convergent."

   (c) Pick your favorite metric space (X, d) and your favorite non-convergent sequence (x_n) in (X, d). Use your definition from part (b) to prove that your sequence (x_n) is not convergent.

2. Let X be a set. We call a function $d : X \times X \to \mathbb{R}$ a *pseudometric* if the following hold:

   (a) For all $x \in X$, we have $d(x, x) = 0$.

   (b) For all $x, y, \in X$, we have $d(x, y) = d(y, x)$.

   (c) For all $x, y, z \in X$ we have $d(x, y) \leq d(x, z) + d(z, y)$.

   Let d be a pseudometric on X. Do the following:

   (a) Explain how this definition differs from the definition of a *metric*.

   (b) Show that $d(x, y) \geq 0$ for all $x, y \in X$.

   (c) Let $x, y \in X$. Show that if $d(x, y) = 0$ then $d(x, z) = d(y, z)$ for all $z \in X$.

   (d) Show that the relation $\sim$ on X given by $x \sim y$ iff $d(x, y) = 0$ is an equivalence relation.

(Additional Space for Work on Problem 2)

3. Show that there does not exist a rational number q such that $q^3 = 2$.

4. Show that

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

for all $n \in \mathbb{N}$.

5. Let ρ be the discrete metric on $\mathbb{R}$, let d be the Euclidean metric on $\mathbb{R}$, and let

$$f : (\mathbb{R}, \rho) \to (\mathbb{R}, d)$$

be the function defined by $f(r) = r$ for all $r \in \mathbb{R}$.

(a) Is f continuous? Prove your answer.

(b) Is f^{-1} continuous? Prove your answer.

6. Let $A \subseteq \mathbb{R}$ be a closed subset of $\mathbb{R}$ in the Euclidean metric. Show that $A \times \{0\} \subseteq \mathbb{R}^2$ is a closed subset of $\mathbb{R}^2$ in the Euclidean metric.

7. Do **ONE** of the following:

   (a) Show that there are infinitely many primes $p \in \mathbb{N}$ such that $p+2$ is also a prime.

   (b) Show that every even $n \in \mathbb{N}$ with $n > 2$ can be written as $n = p+q$ for some primes p and q.

   (c) Let $p \in \mathbb{N}$ be a prime and let $a \in \mathbb{Z}$ be arbitrary. Show that if $a^2 \equiv 1 (\text{mod } p)$ then either $a \equiv 1 (\text{mod } p)$ or $a \equiv -1 (\text{mod } p)$.

8. Write a short ($\leq$ 2 page) essay entitled "Using equivalence relations to construct new mathematical objects." You do not need to include proofs, and your essay should look like an "English class essay," i.e. should be in usual paragraph format. Include examples from class to illustrate your points.

(Additional Room For Solutions)

(Additional Room For Solutions)