Math 1A Quiz 9 Solutions

April 21st, 2008

1. Find the area of the largest rectangle with sides parallel to the coordinate axes which can be inscribed in the ellipse $x^2 + \frac{y^2}{4} = 1$.

Solution: Let (x, y) be the coordinates of the corner of the rectangle which lies in the first quadrant. Then the area of the rectangle is $A = 4xy$. We want to maximize A with respect to the constraint $x^2 + \frac{y^2}{4} = 1$. Solving this constraint for y gives $y = \sqrt{4 - 4x^2} = 2\sqrt{1 - x^2}$ (Note we don't have to worry about signs here since we only care about positive x and y). So we get the formula $A = 4xy = 8x\sqrt{1 - x^2}$.

Now, taking derivatives gives:

$$\begin{aligned} A'(x) &= 8\sqrt{1-x^2} + 8x\frac{1}{2} \cdot \frac{1}{\sqrt{1-x^2}} \cdot (-2x) \\ &= 8\sqrt{1-x^2} - \frac{8x^2}{\sqrt{1-x^2}} \\ &= \frac{8 - 16x^2}{\sqrt{1-x^2}}. \end{aligned}$$

The critical points of $A(x)$ occur either when $A'(x) = 0$ or $A'(x)$ does not exist. These correspond to the roots of the numerator and denominator of A', respectively. The roots of the numerator are at $x = \pm\frac{\sqrt{2}}{2}$ (we only care about the positive solution) and the roots of the denominator are at $x = \pm 1$ (again, we only care about the positive solution).

Now, we have to check the critical points and the endpoints as possible maxima; our endpoints are the largest and smallest acceptable values of x, namely 0 and 1. Since $A(0) = A(1) = 0$, we don't have to worry about these points; the remaining critical point is $x = \frac{\sqrt{2}}{2}$, which must be where our maximum occurs. Plugging back into the formula for A gives

$$A_{\text{max}} = 8\frac{\sqrt{2}}{2} \cdot \sqrt{1 - \left(\frac{\sqrt{2}}{2}\right)^2} = 4.$$

Note: There's a nice trick here that can simplify your life a lot. To maximize A, it's enough to maximize A^2, which has a nicer formula: $A^2 = 64x^2(1 - x^2) = 64x^2 - 64x^4$.

2. Suppose you want to use Newton's Method to approximate $\sqrt[3]{30}$. Show all of the steps you would go through to get the third approximation, x_3.

Solution: Newton's method is for finding roots, so you need a function of which $\sqrt[3]{30}$ is a root; the function $f(x) = x^3 - 30$ is the right choice. A good initial approximation to $\sqrt[3]{30}$ is $\sqrt[3]{27} = 3$, so take $x_1 = 3$. Then just chug through two iterations:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 3 - \frac{3^3 - 30}{3(3)^2}$$

and

$$x_3 = x_2 - \frac{x_2^3 - 30}{3x_2^2}.$$

You could actually calculate these out... but that's what calculators are for.

Note: Some people tried $f(x) = x - \sqrt[3]{30}$ as their function. But this is no good, because you need a function whose values you can compute, and to compute the values of this $f(x)$, you would already need to have computed $\sqrt[3]{30}$!