Math 1A Quiz 8 Solutions

April 4th, 2008

1. a) Let f and g be two functions, and let a be a real number. What two conditions must be satisfied for a to be in the domain of $f \circ g$?

   b) What is the domain of $f(x) = \cosh^{-1}\left(\frac{4}{\pi}\sin^{-1}(x)\right)$?

   Solution to a): As discussed in section, the two criteria to check are:

   (a) a must be in the domain of $g(x)$, and

   (b) $g(a)$ must be in the domain of $f(x)$.

   Solution to b): The domain of $\sin^{-1}(x)$ is $[-1, 1]$, so the domain of $\frac{4}{\pi}\sin^{-1}(x)$ is also $[-1, 1]$. The domain of $\cosh^{-1}(x)$ is $[1, \infty)$. Applying the criterion from part a), the domain of $\cosh^{-1}(\frac{4}{\pi}\sin^{-1}(x))$ are those real numbers a such that:

   (a) a is in the domain of $\frac{4}{\pi}\sin^{-1}(x)$ (so $-1 \leq a \leq 1$), and

   (b) $\frac{4}{\pi}\sin^{-1}(a)$ is in the domain of $\cosh^{-1}(x)$ (so $\frac{4}{\pi}\sin^{1}(a) \geq 1$).

   The second condition says $\sin^{-1}(a) \geq \frac{\pi}{4}$; equivalently, $a \geq \sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$. So the answer is: the domain of $\cosh^{-1}(\frac{4}{\pi}\sin^{-1}(x))$ are those x such that $-1 \leq x \leq 1$ and $\frac{\sqrt{2}}{2} \leq x$. In other words, the domain is $[\frac{\sqrt{2}}{2}, 1]$.

2. Show that $\sqrt{1+x} < 1 + \frac{1}{2}x$ if $x > 0$.

Note: There are many, many solutions to this problem. Here are the best ones I know/saw:

Solution 1: Let $f(x) = \sqrt{1+x}$. Applying the mean value theorem with $a = 0$ and $b = x > 0$ says that there is a c with $0 < c < x$ such that

$$\frac{\sqrt{1+x} - \sqrt{1+0}}{x - 0} = f'(c) = \frac{1}{2\sqrt{1+c}}$$

since $c > 0$, we have $\sqrt{1+c} > 1$, so $\frac{1}{\sqrt{1+c}} < 1$, hence $\frac{1}{2\sqrt{1+c}} < \frac{1}{2}$, so the above equation gives:

$$\frac{\sqrt{1+x} - 1}{x} = \frac{1}{2\sqrt{1+c}} < \frac{1}{2}.$$

Multiplying both sides by x gives $\sqrt{1+x} - 1 < \frac{1}{2}x$, so $\sqrt{1+x} < \frac{1}{2}x + 1$.

Solution 2: It is equivalent to show that $1 + \frac{1}{2}x - \sqrt{1+x} > 0$ for $x > 0$. Let $f(x) = 1 + \frac{1}{2}x - \sqrt{1+x}$. Then $f(0) = 1 + 0 - \sqrt{1} = 0$. Take derivatives; we get:

$$f'(x) = \frac{1}{2} - \frac{1}{2\sqrt{1+x}},$$

and the same argument as in Solution 1 shows that $f'(x) > 0$ for $x > 0$. Now we proceed by a contradiction argument:
Suppose we had $f(b) \leq 0$ for some $b > 0$. Then the mean value theorem says there is a c with $0 < c < b$ such that

$$f'(c) = \frac{f(b) - f(0)}{b - 0} = \frac{f(b)}{b}.$$

Since $f(b) \leq 0$ and $b > 0$, we have $\frac{f(b)}{b} \leq 0$. But $f'(c) > 0$ since $c > 0$, so this is a contradiction. Hence our assumption is wrong, so in fact $f(b) > 0$ for every $b > 0$, which is what we wanted to show.

Note on Solution 2: The same argument shows that in general, if f and g are two differentiable functions and a is some real number, then to show $f(x) < g(x)$ for $x > a$, it is enough to show that $f(a) \leq g(a)$ and $f'(x) < g'(x)$ for $x > a$.

Solution 3: We can do this purely algebraically; first we work backwards: suppose $x > 0$. Then squaring both sides of $\sqrt{x+1} < 1 + \frac{1}{2}x$ gives $x + 1 < (1 + \frac{1}{2}x)^2 = 1 + x + \frac{1}{4}x^2$. Subtract $x + 1$ from both sides; this says $0 < \frac{1}{4}x^2$, which *is* true for $x > 0$.

Now let's use this to prove what we want to prove: for $x > 0$, we have $0 < \frac{1}{4}x^2$, so $1 + x < 1 + x + \frac{1}{4}x^2 = (1 + \frac{1}{2}x)^2$. Take square roots of both sides (we can do this since both sides are positive). We get $\sqrt{1+x} < \sqrt{(1 + \frac{1}{2}x)^2} = \left|1 + \frac{1}{2}x\right| = 1 + \frac{1}{2}x$ (the last equality is true since $x > 0$). This is what we wanted to show.