Math 1A Quiz 7 Solutions

March 21st, 2008

1. Let $f(x) = \tanh^{-1}(\sin x)$.

   a) What is the domain of f?

   b) Find a formula for $f'(x)$. Simplify as much as possible.

   Solution to a): For x to be in the domain of $f(x) = \tanh^{-1}(\sin x)$, we must have that:

   (a) x is in the domain of sin, and

   (b) $\sin x$ is in the domain of $\tanh^{-1}$.

   Since the domain of sin is $\mathbb{R}$, we just need $\sin x$ to be in the domain of $\tanh^{-1}$, which is $(-1, 1)$. So we want $-1 < \sin x < 1$. Since the range of sin is $[-1, 1]$, we just want $\sin x \neq \pm 1$. So the domain is $x \neq \frac{\pi}{2} + n\pi$. We could also write that the domain is

   $$\cdots \cup \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \cup \cdots$$

   Solution to b): We use the identity $\frac{d}{dx} \tanh^{-1}(x) = \frac{1}{1-x^2}$, plus chain rule:

   $$f'(x) = \frac{1}{1 - \sin^2(x)} \cos x.$$

   Since $1 - \sin^2(x) = \cos^2 x$, $f'(x) = \frac{1}{\cos^2(x)} \cos x = \frac{1}{\cos x}$.

2. Find $\lim_{x \to 0} (1 - 2x)^{1/x}$.

   Solution: First write $(1 - 2x)^{1/x} = e^{(1/x)\ln(1-2x)}$. We want to figure out what happens to this exponent as x goes to 0, i.e. we want to find:

   $$\lim_{x \to 0} \frac{\ln(1 - 2x)}{x}.$$

Now, as $x \to 0$, we have $\ln(1-2x) \to \ln 1 = 0$ and $x \to 0$, so we can apply L'Hospital's rule to this limit, and we get:

$$\lim_{x\to 0} \frac{\ln(1-2x)}{x} = \lim_{x\to 0} \frac{\frac{-2}{1-2x}}{1} = \lim_{x\to 0} \frac{-2}{1-2x} = -2.$$

So the exponent in our original expression $e^{(1/x)\ln(1-2x)}$ goes to -2 as x goes to 0. Hence

$$\lim_{x\to 0}(1-2x)^{1/x} = \lim_{x\to 0} e^{(1/x)\ln(1-2x)} = e^{-2}.$$