

Math 1A Quiz 6 Solutions and Grading

March 14th, 2008

1. The half-life of cesium-137 is 30 years. Suppose we have a 100-mg sample.

a) Find the mass that remains after t years.

b) After how long will only 1 mg remain?

Solution to a): We use the equation $y(t) = Ae^{kt}$ and solve for the constants A and k . We're given that $y(0) = 100$ and $y(30) = 100/2 = 50$. Hence:

$$\begin{aligned}100 = y(0) &= Ae^{k \cdot 0} = A \\50 = y(30) &= 100e^{k \cdot 30} \\1/2 &= e^{30k} \\\ln(1/2) &= 30k \\\frac{\ln(1/2)}{30} &= k\end{aligned}$$

So our final equation is $y(t) = 100e^{\frac{\ln(1/2)}{30}t} = 100e^{-\frac{\ln(2)}{30}t}$. Many people had just memorized the formula for the half-life equation, which is also perfectly fine, provided you memorize it *correctly*.

Solution to b): Set $1 = y(t) = 100e^{-\frac{\ln(2)}{30}t}$ and solve for t :

$$\begin{aligned}1/100 &= e^{-\frac{\ln(2)}{30}t} \\\ln(1/100) &= -\frac{\ln(2)}{30}t \\-\ln(100) &= -\frac{\ln(2)}{30}t \\\frac{30 \ln(100)}{\ln(2)} &= t.\end{aligned}$$

I saw this answer in many equivalent forms and they're all fine.

Grading: Part a) was worth 2 points, part b), 1 point. Part a) was pretty much all or nothing. One person made a small algebra error and lost a point, but in every other case, either you got full credit or no credit. For part b), if you got the wrong formula in a) but solved for t correctly in part b) according to your formula, I gave you credit.

2. For each of the following functions f , state whether f has an inverse. If no, explain why not. If yes, find $f^{-1}(4)$.
- a) $f(x) = x^3 - x$.
- b) $f(x) = 4x + \sin(x - 1)$.
- c) $f(x) = \ln(\ln(x))$.

Solution to a): The function $f(x)$ is not one-to-one. Factoring gives: $f(x) = x(x + 1)(x - 1)$, so $f(-1) = f(0) = f(1) = 0$, hence f fails the “horizontal line test.”

Solution to b): The function $f(x)$ is one-to-one; taking the derivative gives $f'(x) = 4 + \cos(x - 1)$. Since $-1 \leq \cos(x - 1) \leq 1$, we have $f'(x) \geq 3$ for all x . So $f(x)$ is strictly increasing, hence one-to-one. We find $f^{-1}(4)$ by guessing well: notice $4 \cdot 1 + \sin(1 - 1) = 4$, so $f^{-1}(4) = 1$.

Solution to c): The function $f(x)$ is invertible because it's a composition of invertible functions. Also, we can find an inverse: $y = \ln(\ln(x))$ gives $e^y = \ln(x)$ gives $e^{e^y} = x$, so the function $g(x) = e^{e^x}$ is an inverse to $f(x)$. Hence $f^{-1}(4) = g(4) = e^{e^4}$.

Grading: Each part was worth 1 point; no partial credit was given on any one part. A lot of people correctly stated that the function in a) was not invertible, but then gave no justification, or stated “ $f(x)$ is not invertible because it is not one-to-one,” which is like saying “cake is tasty because it is delicious.” You need to justify this with evidence! For part b), you did not have to justify that f was one-to-one, though obviously I'm happy if you did.