Math 1A Quiz 3

February 15th, 2008

Name ____________________ SID ________________

1. (Version 1:) Find a degree-3 polynomial $p(x)$ such that the graph of $p(x)$ has a horizontal tangent line at the point $(0, 0)$ and a tangent line with slope 1 at the point $(1, 3)$.

   (Version 2:) Find a degree-3 polynomial $p(x)$ such that the graph of $p(x)$ has a horizontal tangent line at the point $(0, 0)$ and a tangent line with slope 1 at the point $(2, 3)$.

   Solution to Version 2: Let $p(x) = Ax^3 + Bx^2 + Cx + D$. We want to figure out what A, B, C, and D should be in order to have p satisfy the given conditions. Note that $p'(x) = 3Ax^2 + 2Bx + C$. The given conditions tell us four things: the graph of p passes through the points $(0, 0)$ and $(2, 3)$ – so $p(0) = 0$ and $p(2) = 3$ – and the slope of the tangent to the graph of p is 0 for $x = 0$ and 1 for $x = 2$, i.e. $p'(0) = 0$ and $p'(2) = 1$. Each of these conditions gives us an equation involving A, B, C, and D:

$$0 = p(0) = 0A + 0B + 0C + D = D$$
$$0 = p'(0) = 0 \cdot 3A + 0 \cdot 2B + C = C$$
$$3 = p(2) = 8A + 4B + 2C + D$$
$$1 = p'(2) = 4 \cdot 3A + 2 \cdot 2B + C$$

   Using the first two equations (i.e. $C = 0 = D$), we reduce the second two equations to

$$3 = 8A + 4B \qquad \text{and} \qquad 1 = 12A + 4B.$$

   Subtract the first of these equations from the second to find $-2 = 4A$, so $A = -1/2$. Plug this back in to the first equation to get $3 = 8 \cdot (-1/2) + 4B$, so $7 = 4B$, hence $B = 4/7$. So the solution is:

$$p(x) = \frac{-1}{2}x^3 + \frac{4}{7}x^2.$$

Note: When you get your solution to this problem, you should check that it actually is a solution! (So for the above solution, take a minute and check $p(0) = 0$, $p'(0) = 0$, and so on.) This doesn't take long and will save you from making algebra errors.

2. (Version 1:) Let f be the function defined as follows:

$$f(x) = \begin{cases} x^2 \cot(2x) & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$$

Find $f'(0)$, or explain why it does not exist.

(Version 2:) Same as above, with $\cot(2x)$ replaced by $\cot(3x)$.

Solution to Version 1: We use the formula for $f'(0)$:

$$\begin{aligned} f'(0) &= \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} \\ &= \lim_{x\to 0} \frac{x^2 \cot(2x) - 0}{x - 0} \\ &= \lim_{x\to 0} \frac{x^2 \cot(2x)}{x} \\ &= \lim_{x\to 0} x \cot(2x) \\ &= \lim_{x\to 0} \frac{x}{\sin(2x)} \cdot \frac{\cos(2x)}{1} \\ &= \lim_{x\to 0} \frac{1}{2} \cdot \frac{2x}{\sin(2x)} \cdot \frac{\cos(2x)}{1} \\ &= 1/2. \end{aligned}$$

The last step here is using the identity $\lim_{x\to 0} \frac{2x}{\sin(2x)} = 1$ and the continuity of $\cos(2x)$.

How NOT to do it: The product rule does *not* apply here, nor does the chain rule. First of all, our function is not given by $x^2 \cot(2x)$ near 0; it's given by a piecewise formula and you have to take this into account. Second, $\cot(2x)$ is not even defined for $x = 0$, so it's certainly not continuous or differentiable, and hence it doesn't mean anything to write $(\cot(2x))'$ at 0. When you see that a function has weird behavior at one point, your *only* option is to use the limit definition for the derivative. This is also the *only* way to show that a derivative does not exist.