Math 1A Quiz 2 Solutions

February 8th, 2008

1. (Version 1:) Calculate:

$$\lim_{x\to-\infty} \frac{x^4+x^3+2+\sin x}{\sqrt{1+2x^8}+2x^4+3}$$

(Version 2:) Calculate:

$$\lim_{x\to-\infty} \frac{3x^4+x^3+2}{\sqrt{1+2x^8}+2x^4+3+\sin x}$$

Solution: Both versions are solved by the same method: divide both the numerator and denominator by x^4. Here's how version 1 ends up:

$$\lim_{x\to-\infty} \frac{1+\frac{1}{x}+\frac{2}{x^4}+\frac{\sin x}{x^4}}{\frac{1}{x^4}\sqrt{1+2x^8}+2+\frac{3}{x^4}}.$$

Now, most of these terms go to 0 as $x \to -\infty$ since they're just constants over some power of x. Nothing changes about the 1 in the numerator and the 2 in the denominator. The only terms we need to deal with are the $\frac{\sin x}{x^4}$ term and the term involving the square root. Now, $-1 \leq \sin x \leq 1$, so also $\frac{-1}{x^4} \leq \frac{\sin x}{x^4} \leq \frac{1}{x^4}$. As $x \to -\infty$, both $\frac{1}{x^4}$ and $\frac{-1}{x^4}$ go to 0. The Squeeze theorem tells us that also $\frac{\sin x}{x^4}$ goes to 0.

We just need to deal now with the square root term. We have that $\frac{1}{x^4} = \left|\frac{1}{x^4}\right| = \sqrt{\frac{1}{x^8}}$. Hence $\frac{1}{x^4}\sqrt{1+2x^8} = \sqrt{\frac{1}{x^8}+2}$. As x goes to $-\infty$, this clearly goes to $\sqrt{2}$. Putting this all together, we get that the limit is

$$\frac{1+0+0+0}{\sqrt{2}+2+0} = \frac{1}{\sqrt{2}+2}.$$

Important Things: A lot of people claimed in their solution that $\lim_{x\to-\infty} \frac{\sin x}{x} = 1$. This is false. The same Squeeze theorem argument as above shows that $\lim_{x\to-\infty} \frac{\sin x}{x} = 0$. Intuitively, we see that the denominator of this expression gets very large (absolute-value-wise) whereas the numerator stays between -1 and 1, so we should expect that this limit is 0. If you made the above false claim, or didn't justify that $\lim_{x\to-\infty} \frac{\sin x}{x^4} = 0$, you lost a point.

Second important thing: suppose the problem was instead to calculate:

$$\lim_{x\to-\infty} \frac{x^3+2+\sin x}{\sqrt{1+2x^6}+2x^3+3}.$$

As above, we would divide by the highest power (which is x^3; we count powers inside square roots as half). Most of this would go the same, but notice that for $x < 0$, we have $\frac{1}{x^3} = -\left|\frac{1}{x^3}\right| = -\sqrt{\frac{1}{x^6}}$, so we have $\frac{1}{x^3}\sqrt{1+2x^6} = -\sqrt{\frac{1}{x^6}+2}$. Hence the whole limit would be

$$\frac{1+0+0}{-\sqrt{2}+2} = \frac{1}{2-\sqrt{2}}.$$

It's easy to get this sign wrong, so be careful!

Really bad mistakes: Several people think that $\sqrt{a+b} = \sqrt{a}+\sqrt{b}$. This is really, really, really not true. Almost any pair of a and b will give a counterexample. For example, it is definitely not true that $\sqrt{2} = \sqrt{1+1} = \sqrt{1}+\sqrt{1} = 2$. You should not just manipulate formulas randomly. Think about what they mean! None of this is supposed to be miracle algebra; it's just a matter of using the tools you've been given.

Also, I saw one $\frac{\sin x}{x^8} = \sin(1/x^7)$. This is fantastically wrong.

2. Show that the equation $x^4 = e^x + 2$ has at least two solutions. (Remember that e is approximately 2.7).

Solution: This is an intermediate value theorem problem; we use the IVT twice to isolate two roots. So, finding a root of this equation is the same as finding a 0 of the function $f(x) = x^4 - e^x - 2$. This function is continuous since it's a sum of continuous functions. We have that:

$$f(0) = 0 - e^0 - 2 = -3 < 0$$

$$f(2) = 16 - e^2 - 2 \approx 16 - 7 - 2 = 7 > 0$$

and

$$f(-2) = 16 - \frac{1}{e^2} - 2 > 0.$$

So since $f(0) < 0 < f(2)$, the IVT tells us that f has a root in the interval $(0, 2)$, and similarly, since $f(-2) > 0 > f(0)$, the IVT tells us that f has a root in the interval $(-2, 0)$. These intervals don't overlap, so the two roots we've found can't be the same. Hence f has at least two roots.

Note: In fact, f has at least 3 roots; $\lim_{x\to\infty} f(x) = -\infty$, so f dips back below zero at some positive number larger than 2.