Math 1A Quiz 1 Solutions

February 2nd, 2008

Let $f(x)$ be the function:

$$f(x) = \begin{cases} \sin x & , \quad x < \frac{\pi}{2} \\ 3 & , \quad x = \frac{\pi}{2} \\ x - \frac{\pi}{2} + 1 & , \quad x > \frac{\pi}{2} \end{cases}$$

a) Graph this function.

Solution: I don't have the tech knowledge to make this graphic, but the description in words is as follows: to the left of the line $x = \frac{\pi}{2}$, the graph is the graph of $\sin(x)$; if you don't recall what this looks like, go look it up on Wikipedia. At the point $(\frac{\pi}{2}, 1)$, there's an open circle (indicating that this point is not part of our graph, but that the points of $\sin(x)$ approach this point). There's a single point at $(\frac{\pi}{2}, 3)$. To the right of the line $x = \frac{\pi}{2}$, we have a line with slope 1 emanating from the open circle at $(\frac{\pi}{2}, 1)$ (this is the line $y = x - \frac{\pi}{2} + 1$).

Note: If you can't follow any of the above, please see me at office hours and I'll be happy to draw this graph for you.

b) Does $\lim_{x \to \frac{\pi}{2}} f(x)$ exist? If so, what is the value of this limit? If not, explain why not. (You don't need to prove it!)

Solution: The limit exists, and is equal to 1. To check that the limit exists, it is enough to check that $\lim_{x \to \frac{\pi}{2}^+} f(x)$ and $\lim_{x \to \frac{\pi}{2}^-} f(x)$ both exist and are equal. We have:

$$\lim_{x \to \frac{\pi}{2}^+} f(x) = \lim_{x \to \frac{\pi}{2}^+} \sin(x) = 1$$

and

$$\lim_{x \to \frac{\pi}{2}^-} f(x) = \lim_{x \to \frac{\pi}{2}^-} x - \frac{\pi}{2} + 1 = 1.$$

(Each of the above limits can be determined by looking at the graphs of these functions.)

Note: It is not important that $f(\frac{\pi}{2}) = 3$. Remember that $\lim_{x\to a} f(x)$ cannot see what happens at $x = a$.

c) Find a $\delta > 0$ so that whenever $0 < \left|x - \frac{\pi}{2}\right| < \delta$, we get $|f(x) - 1| < \frac{1}{2}$. [Hint: find a δ for each side of $\frac{\pi}{2}$, then take the smaller one.]

Solution: We follow the advice given. We first find a δ_1 so that when $0 < x - \frac{\pi}{2} < \delta_1$ we get $\left|(x - \frac{\pi}{2} + 1) - 1\right| < \frac{1}{2}$. The second inequality simplifies to $\left|x - \frac{\pi}{2}\right| < \frac{1}{2}$. When $0 < x - \frac{\pi}{2}$, this is the same as saying that $x - \frac{\pi}{2} < \frac{1}{2}$, so we should take $\delta_1 = \frac{1}{2}$.

Now we find a δ_2 so that when $-\delta_2 < x - \frac{\pi}{2} < 0$ we get $|\sin(x) - 1| < \frac{1}{2}$. Find this by drawing a graph of $\sin(x)$. We notice that $\sin(x)$ crosses the line $y = \frac{1}{2}$ at the input $x = \frac{\pi}{6}$ and the graph of $\sin(x)$ stays above this line between $x = \frac{\pi}{6}$ and $x = \frac{\pi}{2}$. So having $\frac{\pi}{6} < x < \frac{\pi}{2}$ will work, which is the same as $\frac{\pi}{3} < x - \frac{\pi}{2} < 0$, i.e. we should take $\delta_2 = \frac{\pi}{3}$.

To find the actual δ, we take $\delta = \min(\delta_1, \delta_2)$. Since $\delta_1 = \frac{1}{2} < \frac{\pi}{3} = \delta_2$, we take $\delta = \frac{1}{2}$.

Note: Everything here is better understood in pictures. If you're confused, you should start by drawing a graph of $f(x)$. For each step in the above description, draw where this step appears on your picture, and things should become clearer.