Math 1A Quiz 11 Solutions

May 2nd, 2008

1. Let
$$f(x) = \int_{2x}^{3x} \frac{u^2-1}{u^2+1}\, du.$$
Find $f'(x)$.

Solution 1: We have that
$$f(x) = \int_{2x}^{3x} \frac{u^2-1}{u^2+1}\, du = \int_0^{3x} \frac{u^2-1}{u^2+1}\, du - \int_0^{2x} \frac{u^2-1}{u^2+1}\, du.$$
Applying the fundamental theorem of calculus and the chain rule gives
$$f'(x) = 3 \cdot \frac{(3x)^2-1}{(3x)^2+1} - 2 \cdot \frac{(2x)^2-1}{(2x)^2+1}.$$
(Explicitly, if we write $G(x) = \int_0^x \frac{u^2-1}{u^2+1}\, du$, the FTOC (part 1) tells us that $G'(x) = \frac{x^2-1}{x^2+1}$. The first equation above shows that
$$f(x) = G(3x) - G(2x),$$
so
$$f'(x) = 3G'(3x) - 2G'(2x) = 3 \cdot \frac{(3x)^2-1}{(3x)^2+1} - 2 \cdot \frac{(2x)^2-1}{(2x)^2+1},$$
as above.)

Solution 2: This is essentially the same thing, but maybe easier to see. Let $G(x)$ be an antiderivative for $\frac{x^2-1}{x^2+1}$. By the FTOC (part 2), we have
$$f(x) = \int_{2x}^{3x} \frac{u^2-1}{u^2+1}\, du = G(3x) - G(2x).$$

Now take derivatives of both sides; we get:

$$f'(x) = 3G'(3x) - 2G'(2x) = 3 \cdot \frac{(3x)^2 - 1}{(3x)^2 + 1} - 2 \cdot \frac{(2x)^2 - 1}{(2x)^2 + 1}.$$

2. Let $g(x) = \sqrt{1 - x^2}$, where $-1 \leq x \leq 1$. Find the average value of $g(x)$ on the interval $[-\sqrt{2}/2, \sqrt{2}/2]$. [Hint: to find the integral, draw a picture!]

Solution: By definition, the average value of $g(x)$ on $[-\sqrt{2}/2, \sqrt{2}/2]$ is:

$$\frac{1}{\sqrt{2}/2 - (-\sqrt{2}/2)} \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \sqrt{1 - x^2}\, dx = \frac{1}{\sqrt{2}} \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \sqrt{1 - x^2}\, dx.$$

Now, we don't know an antiderivative for $\sqrt{1 - x^2}$ (though I saw some inventive ones...), but we can calculate this integral using geometry. If you draw a picture, you'll see that this integral is the region bounded above by the semicircle of radius 1 and lying between $-\sqrt{2}/2$ and $\sqrt{2}/2$. If you draw lines from the origin to the two "corners" of this shape which lie on the unit circle, you'll see that you divide the shape up into three pieces: two right triangles with base $\sqrt{2}/2$ and hypotenuse 1, and a third region which is a sector of a circle. Your two right triangles also have height $\sqrt{2}/2$ by the Pythagorean theorem, so they're 45-45-90 triangles. Using this, you can see that the angle of the sector of the circle is also 90, so it's just a quarter of a circle.

Hence the total area is $2 \cdot \frac{1}{2} \left(\frac{\sqrt{2}}{2}\right)^2 + \frac{\pi}{4}$, which is just $\frac{1}{2} + \frac{\pi}{4}$. So

$$\frac{1}{\sqrt{2}} \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \sqrt{1 - x^2}\, dx = \frac{\frac{1}{2} + \frac{\pi}{4}}{\sqrt{2}}.$$

Note: To understand the solution above, you really need to sit down and draw the pictures. I would insert graphics, but it's really hard to do.