The ϵ-δ definition of a limit

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The ϵ-δ definition of a limit is as follows:
We say the *limit of* $f(x)$ *as* x *approaches* a is L, written

$$\lim_{x\to a} f(x) = L,$$

if for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $|x-a| < \delta$, we have $|f(x) - L| < \epsilon$.

The **idea** of a limit is that the values of $f(x)$ get very close to L as x gets very close to a, and this is all the complex definition above really says. Let's break this definition up into pieces and examine each piece until we can see that this is true.

"For all $\epsilon > 0$": says that we fix some positive constant ϵ. We think of this ϵ as some very small distance; we want to bound $f(x)$ within this small distance from L.

"There exists a $\delta > 0$": this δ is how close x has to be to a for $f(x)$ to be within ϵ of L. Note that δ depends on ϵ. This is extremely important; generally, when we are searching for a δ, we write it *in terms of* ϵ.

"Whenever $|x-a| < \delta$": means "when the distance from x to a is less than δ."

"We have $|f(x) - L| < \epsilon$": means "the distance from $f(x)$ to L is less than ϵ.

Let's put this all together now: the whole definition says that if we are given some very small distance ϵ, we can find a small distance δ so that if the distance from x to a is smaller than δ, then the distance from $f(x)$ to L is smaller than ϵ. In more colloquial terms, given any small distance, we can get $f(x)$ *at most* this distance from L by pushing x close enough to a.

How does this definition work in practice?

First let's do a very simple example: Let $f(x) = x$ and $a = 1$. Then I claim that

$$\lim_{x \to 1} x = 1$$

(so $L = 1$ in our original notation).

The first step to showing this is always the same: let $\epsilon > 0$ be arbitrary. In other words, fix some positive constant ϵ. We don't care what it is, but it's not allowed to change for the rest of our computation. We'll write our δ in terms of ϵ.

We want to find a $\delta > 0$ so that for $|x - 1| < \delta$, we get

$$|f(x) - 1| < \epsilon.$$

Since $|f(x) - 1| = |x - 1|$, if we set $\delta = \epsilon$, then for $|x - 1| < \delta$, we have

$$|f(x) - 1| = |x - 1| < \delta = \epsilon,$$

which is what we wanted.

Now, let's try a more involved statement.

Let's let $f(x) = 3x^3$ and $a = 1$. I claim that

$$\lim_{x \to 1} 3x^3 = 3$$

(so $L = 3$ in our original notation).

Again, the first step is always: let $\epsilon > 0$ be arbitrary.

Now, we want to find a δ so that for $|x - 1| < \delta$, we get

$$|3x^3 - 3| < \epsilon.$$

Dividing this inequality by 3, we get:

$$|x^3 - 1| < \frac{\epsilon}{3},$$

and it is just as good to try to find a δ making this second inequality true. Now, we factor $x^3 - 1$ and we have:

$$|x^3 - 1| = |x - 1||x^2 + x + 1|.$$

Our δ is allowed to be as small as we like, provided it's not zero. Let's declare right now that our δ will be no larger than 1, i.e. $\delta \leq 1$, so that if $|x-1| \leq \delta$, then

$$|x-1| \leq 1,$$

and hence $0 \leq x \leq 2$.

The function x^2+x+1 is increasing for $x \geq 0$, so if $0 \leq x \leq 2$, then $x^2+x+1 \leq 2^2+2+1 = 7$. Hence if $|x-1| \leq 1$, then

$$|x^3-1| = |x-1||x^2+x+1| \leq 7|x-1|.$$

Since we want $|x^3-1| \leq \frac{\epsilon}{3}$, we see it is enough to choose δ so that $7|x-1| \leq \frac{\epsilon}{3}$, which is the same as

$$|x-1| \leq \frac{\epsilon}{21}.$$

So we can take $\delta = \frac{\epsilon}{21}$. There's just one catch! We also needed $\delta \leq 1$ to use our nice trick above, so if $\frac{\epsilon}{21}$ is bigger than 1, we ought to take $\delta = 1$ instead. Hence our final answer is:

$$\delta = \min\{\frac{\epsilon}{21}, 1\}.$$

Let's check that this works. If $|x-1| \leq \delta$, then $|x-1| \leq 1$ and $|x-1| \leq \frac{\epsilon}{21}$, so we have:

$$|3x^3-3| = 3|x^3-1| = 3|x-1||x^2+x+1| \leq 21|x-1| \leq 21\frac{\epsilon}{21} = \epsilon.$$

Note that we actually got the "non-strict" inequality $\leq$ here. It doesn't matter; if we take a new ϵ_2 so that $0 < \epsilon_2 < \epsilon$, then we have shown that there exists a $\delta_2 > 0$ so that if $|x-1| \leq \delta_2$, then

$$|3x^3-3| \leq \epsilon_2 < \epsilon,$$

so we can get a "strict" inequality $<$ this way.