Math 1A Worksheet 9

September 19th, 2007

1. Give examples of the following:

   a) Two functions f and g such that both f and g are continuous nowhere, but $f + g$ is continuous everywhere.

   b) Two everywhere continuous functions f and g such that f and g are both not differentiable at 0, but $f + g$ is differentiable at 0.

   c) Two functions f and g such that $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ exist, but
   $$\lim_{x\to 0}(f \circ g)(x)$$
   does not.

   d) Two functions f and g such that $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to\infty} g(x) = -\infty$, and
   $$\lim_{x\to\infty}(f \circ g)(x) = 0.$$

   Solutions:

   a) Take
   $$f(x) = \begin{cases} 1 & , \quad x \text{ rational} \\ 0 & , \quad x \text{ irrational} \end{cases}$$
   and
   $$g(x) = \begin{cases} 0 & , \quad x \text{ rational} \\ 1 & , \quad x \text{ irrational.} \end{cases}$$

   b) Take $f(x) = |x|$, $g(x) = -|x|$.

   c) Many, many possibilities. For example, $f(x) = \frac{1}{x-1}$ and $g(x) = 1 + x$.

d) Take $f(x) = e^x$ and $g(x) = -x$. Here g could be replaced by *any* function with

$$\lim_{x \to \infty} g(x) = -\infty.$$

2. Let $f(x) = \sqrt[3]{x}$. Use the $x - a$ formula for the derivative to find the $f'(a)$ when $a \neq 0$. The same calculation should show that $f'(0)$ does not exist. Draw a graph of f and attempt to explain why this is the case.
[Note: This is hard, but it will be easier if you remember that $b^3 - c^3 = (b-c)(b^2 + bc + c^2)$.]

Solution: We want to calculate

$$f'(a) = \lim_{x \to a} \frac{x^{1/3} - a^{1/3}}{x - a}.$$

Multiply top and bottom by $x^{2/3} + x^{1/3}a^{1/3} + a^{2/3}$ to get

$$\lim_{x \to a} \frac{x - a}{(x-a)(x^{2/3} + x^{1/3}a^{1/3} + a^{2/3})} = \lim_{x \to a} \frac{1}{x^{2/3} + x^{1/3}a^{1/3} + a^{2/3}}.$$

Provided a does not equal 0, taking the limit gives:

$$f'(a) = \frac{1}{3a^{2/3}}.$$

If a equals zero, then the numerator of the above limit goes to 1 while the denominator goes to 0, so the limit does not exist, i.e. $f'(0)$ does not exist. Drawing a graph shows that the tangent line to $\sqrt[3]{x}$ at 0 is vertical, so its slope is undefined.