Math 1A Worksheet 8

September 17th, 2007

1. The following limit is the derivative of some function $f(x)$ at the point a:
$$\lim_{h \to 0} \frac{\frac{1}{\sqrt{h+1}} - 1}{h}.$$
What are f and a? Rewrite this derivative in the form
$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$
Find the limit (of either expression).

2. Let $g(x) = \frac{1}{x}$. Find $g'(x)$. Where is g differentiable? What is the domain of g'? Why are these two sets the same?

3. Consider the function
$$f(x) = \begin{cases} 1 & , \quad x < 0 \\ x^2 + 1 & , \quad x \geq 0. \end{cases}$$
Is f differentiable at 0? Draw a picture explaining this.

   Now consider the function
$$g(x) = \begin{cases} 1 & , \quad x < 0 \\ x + 1 & , \quad x \geq 0. \end{cases}$$
Is g differentiable at 0? Explain with both calculations and a picture.

   Finally, consider
$$h(x) = \begin{cases} x^2 + 1 & , \quad x \neq 1 \\ 2 & , \quad x = 1. \end{cases}$$
Is h differentiable at 1? Why can we not differentiate each piece like we could with f and g?

4. Use the intermediate value theorem to show that every polynomial of degree 3 has a real zero, i.e. if $f(x) = ax^3 + bx^2 + cx + d$ with $a \neq 0$, show there is some real number p such that $f(p) = 0$. Give an example of a degree-2 polynomial with no real zeros. Generalize this to show that for any positive integer n, there is a polynomial of degree $2n$ with no real zeros.

   [Challenge Problem: Adapt the above trick to show that every polynomial of odd degree has a real zero, i.e. if $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with n odd and a_n nonzero, show there is some real number d such that $f(d) = 0$. (A heuristic argument is fine. To prove this carefully, you will probably need to know the quotient rule for derivatives.)]

5. Given five points in the unit square, show there is some pair of these points whose distance apart is less than or equal to $\frac{\sqrt{2}}{2}$.