Math 1A Worksheet 5: Solution to Exercise 3

September 10th, 2007

1. The greatest integer function $[[x]]$ is defined by: for each real number a, $[[a]]$ is the largest integer n such that $n \leq a$. So, for example, $[[3/2]] = 1$, since $1 \leq 3/2$ but $2 > 3/2$.

   a) Let n be an integer. What is

   $$\lim_{x \to n^+} [[x]]?$$

   What about the limit from below? Does

   $$\lim_{x \to n} [[x]]$$

   exist?

   b) Now, let a be a real number that is not an integer. Does

   $$\lim_{x \to a} [[x]]$$

   exist? What is it?

   c) From a) and b), for which real numbers is $[[x]]$ continuous?

   d) Calculate

   $$\lim_{x \to 1} \frac{x^2 - \frac{x}{2} - \frac{1}{2}}{x - 1}.$$

   [Hint: the numerator factors!]

e) Find

$$\lim_{x\to 1}\left[\left[\frac{x^2-\frac{x}{2}-\frac{1}{2}}{x-1}\right]\right].$$

Solution:

a) The limit from above exists and is n: if $n \le x < n+1$, then $[[x]] = n$, so $|n - [[x]]| = |n-n| = 0$ for x in this range. In terms of the definition of the limit from above, this says that $\delta = 1$ gives $|n - [[x]]| < \epsilon$ for every epsilon.

The limit from below also exists and is equal to $n-1$. The same argument applies: for x less than but near to n, $[[x]] = n-1$. Since the two one-sided limits of $[[x]]$ at n are not equal,

$$\lim_{x\to n}[[x]]$$

does not exist. Hence in particular $[[x]]$ is not continuous at n.

b) The same type of argument as in part a) shows that if a is not an integer, then

$$\lim_{x\to a^+}[[x]] = [[a]] = \lim_{x\to a^-}[[x]].$$

This says that $[[x]]$ is continuous at a.

c) Parts a) and b) together say that $[[x]]$ is continuous for all non-integer real numbers.

d) This is just the usual factorization trick. We have:

$$\lim_{x\to 1}\frac{x^2-\frac{x}{2}-\frac{1}{2}}{x-1} = \lim_{x\to 1}\frac{(x-1)(x+1/2)}{(x-1)} = \lim_{x\to 1}(x+1/2) = \frac{3}{2}.$$

e) Write

$$g(x) = \frac{x^2-\frac{x}{2}-\frac{1}{2}}{x-1}.$$

By part b), $[[x]]$ is continuous at $3/2 = \lim_{x\to 1} g(x)$. By Theorem 7 on Stewart p. 51 (with $f(x) = [[x]]$, $a = 1$, and $b = 3/2$), we have:

$$\lim_{x\to 1}[[g(x)]] = [[\lim_{x\to 1} g(x)]] = [[3/2]] = 1.$$