Math 1A Worksheet 22: Problem 2 Solution

October 29th, 2007

Problem: Using the MVT, show that $\sqrt{1+x} < 1 + \frac{1}{2}x$ for $x > 0$. [Hint: consider the function $f(x) = 1 + \frac{1}{2}x - \sqrt{1+x}$.]

Solution: Use the given f. We want to show that $\sqrt{1+x} < 1 + \frac{1}{2}x$ for $x > 0$. This is the same as showing that $f(x) > 0$ for $x > 0$. Now, since f is continuous, it suffices to show that $f(a) > 0$ for *some* $a > 0$ and that f never crosses the positive x-axis. (Draw a picture to convince yourself!)

The first condition is easy to verify: $f(3) = 1/2 > 0$, so $a = 3$ is a good choice. We rephrase the second condition: we want to show that there is no $b > 0$ such that $f(b) = 0$. We now do a proof by contradiction.

Suppose there is some $b > 0$ such that $f(b) = 0$. Then $f(0) = 0 = f(b)$, and so Rolle's Theorem (or the MVT) implies that there is some c with $0 < c < b$ such that $f'(c) = 0$. But

$$f'(x) = \frac{1}{2}\left(1 - \frac{1}{\sqrt{1+x}}\right).$$

For $x > 0$, we have $\sqrt{1+x} > 1$, so $\frac{1}{\sqrt{1+x}} < 1$ and hence $1 - \frac{1}{\sqrt{1+x}} > 0$. Hence $f'(x) > 0$ for all $x > 0$, and so no such c can exist. This is a contradiction, hence no such b can exist and we are done.