Math 1A Worksheet 10

September 24th, 2007

1. Let
$$f(x) = \left\{ \begin{array}{lcl} x^2 \sin\left(\frac{1}{x}\right) & , & x \neq 0 \\ 0 & , & x = 0 \end{array} \right\}.$$

   Find $f'(x)$. Where is $f'(x)$ continuous?

   Solution: For $a \neq 0$, we can find $f'(a)$ by using our usual rules for differentiation; we get
$$f'(a) = 2a \sin\left(\frac{1}{a}\right) - \cos\left(\frac{1}{a}\right).$$

   For $a = 0$, we **cannot** use these rules, because $\sin(1/x)$ is not even defined (much less differentiable) at 0. We have to use the limit definition of the derivative:
$$f'(0) = \lim_{x \to 0} \frac{x^2 \sin\left(\frac{1}{x}\right) - 0}{x - 0} = \lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0.$$

   So our overall calculation showed:
$$f'(x) = \left\{ \begin{array}{lcl} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & , & x \neq 0 \\ 0 & , & x = 0. \end{array} \right.$$

   Note now that $\lim_{x \to 0} f'(x)$ does not even exist, since $2x \sin(1/x)$ goes to 0 while $\cos(1/x)$ fluctuates between -1 and 1 rapidly. Hence f' cannot be continuous at 0.

   Note: The importance of this exercise is that it provides a counterexample to something we thought was true before this; namely, f is an example of a function which is differentiable everywhere, but whose derivative has a point of discontinuity.