Math 1A Quiz 6 Solutions

December 4th, 2007

Name ____________________ SID ________________

1. Let $f(x)$ be a continuous function and a be a fixed positive real number. Define
$$F(x) = \int_a^{x^2} f(t)\,dt.$$

a) Find $F'(x)$.

Solution: Write $H(x) = \int_a^x f(t)\,dt$. Then $H'(x) = f(x)$ by the FTC. And $F(x) = H(x^2)$, so $F'(x) = H'(x^2)2x = f(x^2)2x$.

b) Suppose G is another function such that
$$G'(x) = F'(x) \quad \text{and} \quad G(\sqrt{a}) = 2.$$

Find $F(x) - G(x)$.

Solution: Since $G'(x) = F'(x)$, the functions F and G differ by a constant, i.e. $F(x) - G(x) = c$ for some constant c. Plug in $\sqrt{a}$ for x here; we get $F(\sqrt{a}) - G(\sqrt{a}) = c$. Now,
$$F(\sqrt{a}) = \int_a^{\sqrt{a}^2} f(t)\,dt = \int_a^a f(t)\,dt = 0,$$
so $c = F(\sqrt{a}) - G(\sqrt{a}) = 0 - 2 = -2$. So the answer is $F(x) - G(x) = -2$.

2. Find
$$\int_0^{\sqrt{2}/2} (\sqrt{1-x^2} - x)\,dx$$
by interpreting the integral as an area.

Solution: The integral is the area under the semicircle of radius 1 and above the line $y = x$ between 0 and $\sqrt{2}/2$ (note that $(\sqrt{2}/2, \sqrt{2}/2)$ is the point where the semicircle and the line $y = x$ meet). This is just an eighth of a circle of radius 1, so the integral is one-eighth of the area of a circle of radius 1, i.e. $\pi/8$.

3. Using $\sum$ notation, write a Riemann sum approximating the integral

$$\int_2^4 \sqrt{\cos x}\, dx.$$

Use a partition into n intervals of equal length, and use the *right-hand* endpoints of the intervals as your x_i^*'s. You do NOT need to evaluate this integral.

Solution: To set this up, you should use one of our picture-drawing methods to draw a generic piece of the partition. Since the interval $[2, 4]$ has length 2, we partition into n equal pieces of length $\frac{2}{n}$ (so $\Delta x_i = \frac{2}{n}$ for all i). The ith piece of this partition has right-hand endpoint $2 + \frac{2i}{n}$, so $x_i^* = 2 + \frac{2i}{n}$, and our Riemann sum is just

$$\sum_{i=1}^{n} \Delta x_i f(x_i^*) = \sum_{i=1}^{n} \frac{2}{n}\sqrt{\cos\left(2 + \frac{2i}{n}\right)}.$$

Note: This integral actually doesn't exist or make any sense; the function $\sqrt{\cos x}$ is undefined when $\cos x$ is negative, which is true for all x in $[2, 4]$.

Bonus Point: Let k be a fixed real number. Find all continuous functions f such that $f(x) = \int_0^x kf(t)\, dt$.

Solution: Since f is continuous, $\int_0^x f(t)\, dt$ is differentiable by the FTC, so $f(x) = k\int_0^x f(t)dt$ is also differentiable. Take derivatives of both sides; we get $f'(x) = kf(x)$ (again by the FTC). This says that f is a solution to $y' = ky$; hence $f(x) = Ae^{kx}$. Now, we have

$$A = Ae^0 = f(0) = \int_0^0 kf(t)\, dt = 0,$$

so $f(x) = 0e^{kx} = 0$ is the only solution.