

Math 1A Quiz 5 Solutions

November 19th, 2007

1. A particle is moving with acceleration $a(t) = e^{-t}$ meters per second squared. Its position at time $t = 0$ is $s(0) = 0$ and its average velocity between times $t = 0$ and $t = 1$ is $-1 - 1/e$ meters per second. Find the velocity of the particle at time $t = 0$.

Solution: Antidifferentiating twice, we have $v(t) = -e^{-t} + C$ and $s(t) = e^{-t} + Ct + D$. The average velocity of the particle between times 0 and 1 is just $\frac{s(1)-s(0)}{1-0} = s(1) - s(0)$. We have

$$-1 - \frac{1}{e} = s(1) - s(0) = \frac{1}{e} + C + D - (1 + D) = \frac{1}{e} + C - 1,$$

from which we get

$$C = -\frac{2}{e}.$$

Hence

$$v(0) = -1 + C = -1 - \frac{2}{e} \text{ meters per second.}$$

Note: Remember that the average velocity is total change in position over total change in time! If you get confused by these things, put it into the language of driving so that it's more familiar: if you drive 150 miles in 3 hours, what's your average velocity? It's 50 miles per hour; that's total change in position over total change in time. Also, checking units will help you avoid mistakes. A lot of people had some formula involving velocity in the numerator and time in the denominator; the units in this case would be meters per second **squared**, which doesn't make sense as a value for average velocity.

2. You are going to make a box with a square base and no top. If the surface area of the resulting box is going to be 100 square inches, what is the largest volume of box you can make? [Note: the answer isn't pretty.]

Solution: Let x be the length of one of the sides of the base of the box and h be the height of the box. Then the surface area of the box is $x^2 + 4hx$. Setting this equal to 100 and solving, we get

$$h = \frac{100 - x^2}{4x}.$$

Now, the volume of the box is x^2h , which we can now write totally in terms of x using the above formula, and we get

$$V(x) = 25x - x^3/4.$$

To find the critical points of $V(x)$, we set $V'(x) = 0$ and solve. We get that the critical points occur at $x = \pm 10/\sqrt{3}$. Since a negative length makes no sense, we discard the negative value. Now, the "endpoints" of the interval we are considering correspond to the cases when the box has zero volume, so this critical point must be an absolute maximum. Hence to find the largest possible volume, we plug in, and find:

$$V(10/\sqrt{3}) = 25\frac{10}{\sqrt{3}} - \frac{1000}{12\sqrt{3}}.$$

We could simplify more, but I'm perfectly happy with this answer.

3. Show that for all real numbers a and b ,

$$|\cos a - \cos b| \leq |a - b|$$

Solution: We've done this many times with sin in place of cos. Use the mean value theorem: there's a c between a and b such that

$$\frac{\cos a - \cos b}{a - b} = -\sin c.$$

Note that it doesn't really matter whether a or b is larger, since the formula on the left hand side of this equation will be equal in either case. Multiplying through by the denominator, we get

$$\cos a - \cos b = (-\sin c)(a - b).$$

Taking absolute values, we get:

$$|\cos a - \cos b| = |-\sin c| |a - b|.$$

Now, since $-1 \leq -\sin c \leq 1$, we have $0 \leq |-\sin c| \leq 1$. Hence multiplying by $|-\sin c|$ can only shrink the positive number $|a - b|$, i.e.

$$|\cos a - \cos b| = |-\sin c| |a - b| \leq |a - b|.$$

Bonus Point: Find two functions f and g , defined for all real numbers, such that f and g are both increasing, but fg is decreasing.

Solution: For example, take $f(x) = e^{2x}$ and $g(x) = -e^{-x}$.