Math 1A Quiz 4

October 26th, 2007

Name ____________________ SID ________________

1. Show that $\sin^{-1}(\tanh(x)) = \tan^{-1}(\sinh x)$. [Hint: draw a right triangle with $\sin^{-1}(\tanh x)$ as one of its angles!]

   Solution: Follow the hint! We draw a right triangle with angle $\theta = \sin^{-1}(\tanh x)$ by taking $o = \sinh x$ and $h = \cosh x$. Then $a = \sqrt{\cosh^2 x - \sinh^2 x} = 1$, so $\tan\theta = (\sinh x)/1 = \sinh x$. Since θ is an angle between $-\pi/2$ and $\pi/2$ with $\tan\theta = \sinh x$, we have $\theta = \tan^{-1}\sinh x$, and we are done.

2. Find all x such that $\cos^{-1}(\cos x) = \pi/3$. [A second version of the quiz has cos replaced by sin in both instances.]

   Solution: Note that the domain of $\cos^{-1}(\cos x)$ is all real numbers, since $\cos x$ has domain all real numbers and any output of cos is a valid input for $\cos^{-1}$. Saying $\cos^{-1}(\cos x) = \pi/3$ is the same as saying $\cos x = \cos \pi/3$. This holds for $x = \pi/3 + 2\pi k$ where k is any integer, and also for $x = -\pi/3 + 2\pi k$ where k is any integer. For $\sin^{-1}(\sin(x))$, the method is the same, but we get $x = \pi/3 + 2\pi k$ or $x = 2\pi/3 + 2\pi k$.

3. Find

$$\lim_{x\to 0} \frac{e^{-1/x^2}}{x}$$

   Solution: Rewrite the limit as

$$\lim_{x\to 0} \frac{1/x}{e^{1/x^2}}.$$

   As $x \to 0$, we have a slight issue in that $1/x$ goes to either ∞ or $-\infty$, depending on whether we approach from the left or right. Say we take the limit from above, i.e. we look at

$$\lim_{x\to 0^+} \frac{1/x}{e^{1/x^2}}.$$

Then $1/x \to \infty$ and $e^{1/x^2} \to \infty$, so we can apply l'Hospital's rule. Since $\frac{d}{dx} 1/x = -x^{-2}$ and $\frac{d}{dx} e^{1/x^2} = -2x^{-3} e^{1/x^2}$, we get that our limit is equal to

$$\lim_{x \to 0^+} \frac{-x^{-2}}{-2x^{-3} e^{1/x^2}} = \lim_{x \to 0^+} \frac{x e^{-1/x^2}}{2}.$$

As $x \to 0^+$, both $x \to 0$ and $e^{-1/x^2} \to 0$, so the whole limit is 0. The same argument gives exactly the same limit from below. Hence our original limit is 0.

Cautionary Point: Notice that we could also have tried l'Hospital's rule on the original expression. But beware! This will make the limit not less, but **more** complicated! This is very important to keep in mind for using l'Hospital's rule.

Bonus Point: Find two functions f and g, defined for all real numbers, such that f and g are both invertible, but fg and $f + g$ are both not invertible. (Worth 2 points, one for this quiz, and one applicable to *any* quiz!)

One solution: $f(x) = x$, $g(x) = -x$. Then $fg(x) = -x^2$ is not invertible, and $f(x) + g(x) = 0$ is not invertible.